Quarto!

Wouter M. Koolen

Cakes Talk
Thursday 29th September, 2011

Goals of this talk

- Become a departmental celebrity.
- Serve ~~Dutch stroopwafels~~ Belgian cookies.
- Popularise *Quarto!*
- Legitimise hobby project.
- Fun and empowering toolbox:
    - Combinatorial game theory
    - Academic programming
- Nice example of brain vs computational power:
    - Thought-assisted combinatorial search
    - Combinatorial-search-assisted thought
- Fascinating symmetries

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Outline

1. Quarto crash course

2. The value of Quarto

3. Playing Optimally

Rules: the pieces and Quarto

- The **pieces** are the 16 realisations of four binary properties:

$$\underbrace{\{\text{dark}, \text{light}\}}_{\text{colour}} \times \underbrace{\{\text{tall}, \text{short}\}}_{\text{height}} \times \underbrace{\{\text{round}, \text{square}\}}_{\text{shape}} \times \underbrace{\{\text{hollow}, \text{solid}\}}_{\text{consistency}}$$

- Four pieces form **Quarto** if they agree on a property.

$$Q\{p, q, r, s\} \qquad \text{iff} \qquad p_i = q_i = r_i = s_i \quad \text{for some property } i$$

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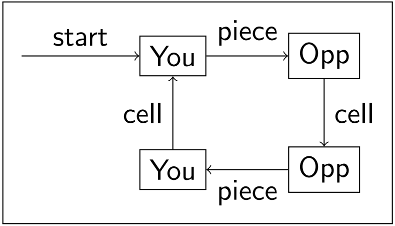
Rules: board, turns and winning

- The **board** has 4×4 cells. Initially empty. Pieces are put aside.
- The game proceeds in **rounds**. Each round has two **plies**:
  - One player gives an unused piece to the other player.
  - The other player places that piece on an empty cell.



- **Win** by forming Quarto in a row, column or (co)diagonal.
- **Draw** when all pieces placed without Quarto.

Questions

- What is the *value of the game*? (i.e. when both players play optimally, does the starting player win, lose or draw?)
- How to play the optimal strategy?

Outline

Naive approach

$$\max_{p_1} \min_{c_1} \min_{p_2} \max_{c_2} \max_{p_3} \ldots \min_{p_{16}} \max_{c_{16}} V(p_1 c_1 \cdots p_{16} c_{16})$$

where

$$V(p_1 c_1 \cdots p_{16} c_{16}) = \begin{cases} -\infty & \text{You disobeyed the rules} \\ -1 & \text{You lose} \\ 0 & \text{Game is a draw} \\ +1 & \text{You win} \\ +\infty & \text{Opp disobeyed the rules} \end{cases}$$

Only $16^{32} \approx 3.4 \cdot 10^{38}$ operations.

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Only $(16!)^2 \approx 4.4 \cdot 10^{26}$ when enforcing the rules.

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Way too many. Q: Any ideas?

Exploiting positionality

In Quarto, the moves from and payoffs in any state depend only on the current position, and not on how the players got there.

function $\textsc{Val}(b)$
    if $\textsc{isQ}(b)$ **return** WIN
    if $\textsc{isFull}(b)$ **return** DRAW
    if we stored that b has value v then **return** v
    if b has given piece p **then**
        $v \leftarrow \max_{c \in cells(b)} \textsc{Val}(b[p@c])$
    else
        $v \leftarrow \max_{p \in pieces(b)} -\textsc{Val}(b \oplus p)$
    end if
    store that b has value v
    return v
end function

We now need $9.9 \cdot 10^{16}$ operations. Still no cigar.

Exploiting symmetries

Some positions are *equivalent*. It suffices to evaluate only one member of each equivalence class.

- Piece symmetries
- Board symmetries

Piece symmetries

Definition (Piece Symmetry)

A *piece symmetry* is a mapping of the 16 pieces to the 16 pieces that preserves Quarto's.

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Q: Find piece symmetries

Fact

There are $4!\,2^4 = 384$ *piece symmetries.*

- *the* 4 *properties can be reordered arbitrarily*
- *the* 2 *values of each property can be flipped*

Board s

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A *board symmetry* is a mapping of the 16 board cells to the 16 board cells that preserves Quarto's.

A board symmetry must map rows/columns to rows/columns and (co)diagonals to (co)diagonals.

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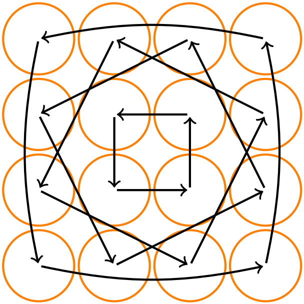
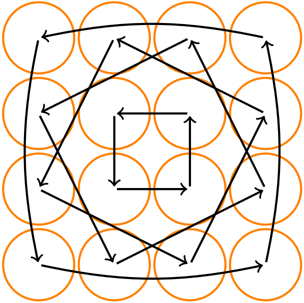
Q: Find board symmetries

Finding board symmetries



counter clockwise rotation

Finding board symmetries



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Q: What about clockwise rotation?

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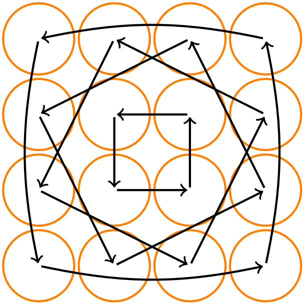

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Q: What about clockwise rotation? A: Rotate ccw thrice

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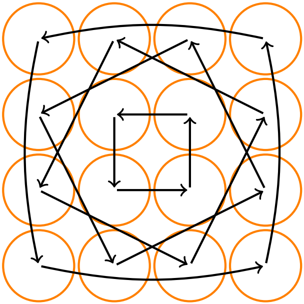

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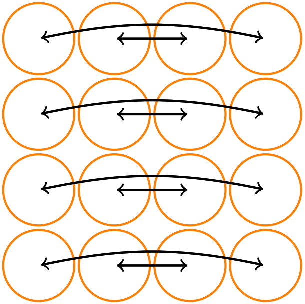

mirror over vertical axis

Q: What about clockwise rotation? A: Rotate ccw thrice

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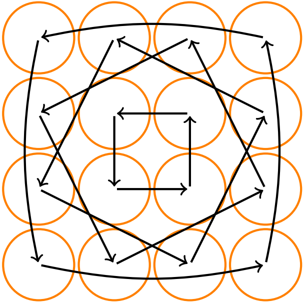

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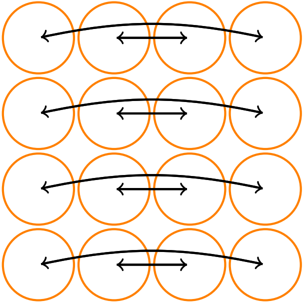

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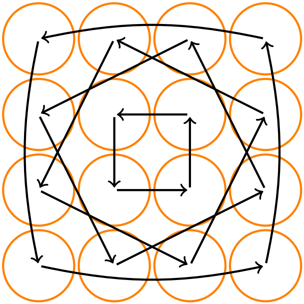
Q: What about clockwise rotation? A: Rotate ccw thrice

Q: Mirror over diagonal?

Finding board symmetries



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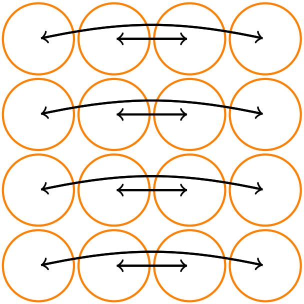
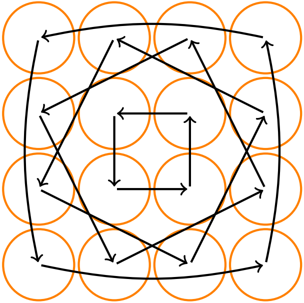


mirror over vertical axis

Q: What about clockwise rotation? A: Rotate ccw thrice

Q: Mirror over diagonal? A: rotate cw, then mirror

Finding board symmetries



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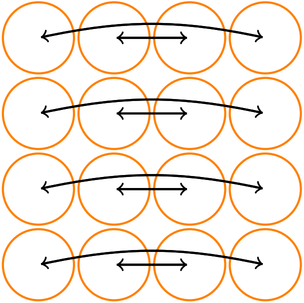

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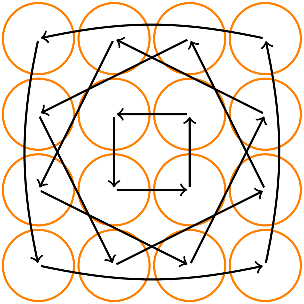
Q: Are there other board symmetries?

Finding board symmetries



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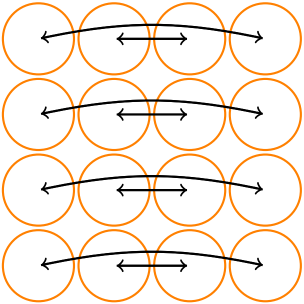

mirror over vertical axis

Q: What about clockwise rotation? A: Rotate ccw thrice

Q: Mirror over diagonal? A: rotate cw, then mirror

Q: Are there other board symmetries?

Q: How to even approach such a question?

Exhaustive enumeration

```
procedure ENUM_SYM(M)
    if M violates group structure then return
    if |M| = 16 then
        print M
    else
        choose a free source cell i
        for each free target cell j do
            ENUM_SYM(M[i → j])
        end for
    end if
end procedure
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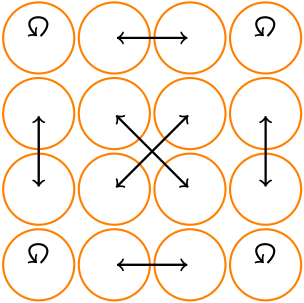
Fact

There are 32 board symmetries.

Finding board symmetries (ctd)



mid flip

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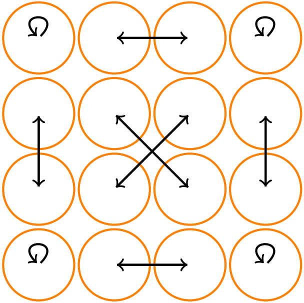

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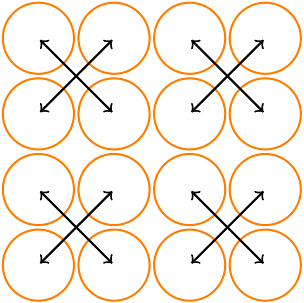
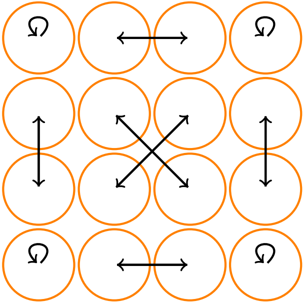
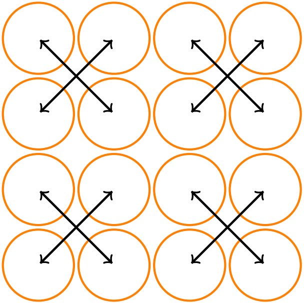


inside out

Finding board symmetries (ctd)



mid flip

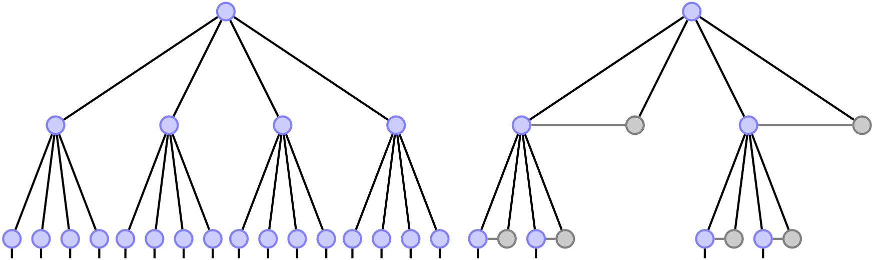


inside out

[Goo04] found 16 (inside out), and [Bro05] found 16(mid flip).

Symmetry benefits

Identifying d positions may *divide* the degree by d. Exponential gain.



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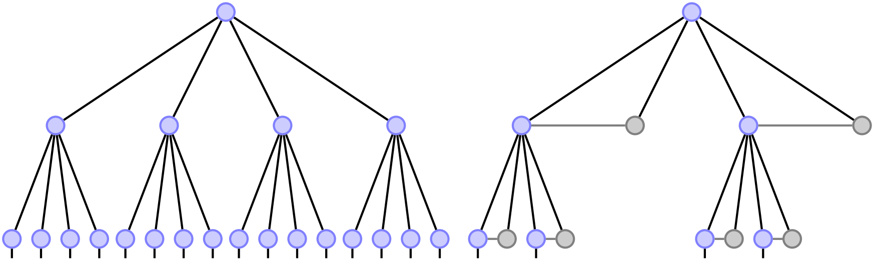
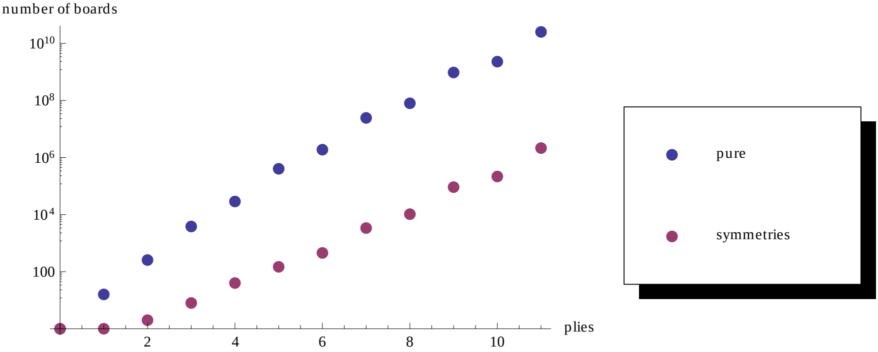


Q: Will it always? If not, what is the least-favourable reduction?

Symmetry benefits ctd.



Good, but we need to explore 32 plies!

How to exploit symmetries

CANONISE picks a canonical representative of each equivalence class.

function VAL(b)
    if ISQ(b) **return** WIN
    if ISFULL(b) **return** DRAW
    $b \leftarrow$ CANONISE(b)
    if we stored that b has value v then **return** v
    if b has given piece p **then**
        $v \leftarrow \max_{c \in cells(b)} \text{VAL}(b[p@c])$
    else
        $v \leftarrow \max_{p \in pieces(b)} -\text{VAL}(b \oplus p)$
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The final trick

Alpha-beta pruning. Rule of thumb: explores only the square root of the original number of positions.

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Fact

The value of Quarto is draw.

Software finds out in 147 minutes on this laptop.

Outline

Implementing the optimal strategy

- So far, we computed the value of the empty board.
- But to play, we need to evaluate any board.
- We can evaluate positions at ≥ 10 plies from scratch in < 5 seconds.
- There are only 106 156 distinct positions at < 10 plies.
- Compute, once and for all, the value of *all of them*.
- Took about 2 weeks on this laptop.
- Now we can evaluate any position fast.
- To play, choose the child that evaluates to the value of the parent.

Kevin S. Brown.
414298141056 quarto draws suffice!
http://www.mathpages.com/home/kmath352.htm, June 2005.

Luc Goossens.
Quarto.
http://web.archive.org/web/20041012023358/http://ssel.vub.ac.be/Members/LucGoossens/quarto/quartotext.htm, October 2004.