Credits (Alphabetical)

Part 1
- A. Philip Dawid
- Steven de Rooij
- Peter Grünwald
- Wouter M. Koolen
- Glenn Shafer
- Alexander Shen
- Nikolai Vereshchagin
- Vladimir Vovk

Part 2
- Wouter M. Koolen
- Vladimir Vovk
Example: Price of Kodak traded on NYSE ’62 – ’06
At a high level

Worst-case guarantees for finance
At a high level

Worst-case guarantees for finance

Even if prices are determined by an adversary we can still achieve interesting financial goals
At a high level

Worst-case guarantees for finance

Even if prices are determined by an adversary we can still achieve interesting financial goals

Goals in this talk

1. Sell high
2. Buy low, sell high
At a high level

Worst-case guarantees for finance

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Goals in this talk

1. Sell high
2. Buy low, sell high

There is some overhead for not knowing the perfect trading time(s)
At a high level

Worst-case guarantees for finance

Even if prices are determined by an adversary we can still achieve interesting financial goals

Goals in this talk

1. Sell high
2. Buy low, sell high

There is some overhead for not knowing the perfect trading time(s)

We characterize the achievable overheads (using a surprisingly elegant formula)

We find a *canonical representation* of achievable guarantees.
1. Introduction

2. Sell high

3. Buy low, sell high
Goal

We start with capital 1$
We consider trading in a single security (with initial price 1$/share)
We want to become rich when the share price is *ever high*

A financial expert claims to have a secret strategy that will accomplish our goal. She shows us a function $F$, and guarantees to keep our capital above $F(y)$ for all exceeded price levels $y$.

Ideally, $F(y)$ is close to $y$.

We would like to find out:

- Is guaranteeing $F$ possible?
- Can more than $F$ be guaranteed?
- Can we reverse engineer a strategy for $F$?
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Example guarantees $F$

![Graph showing guaranteed capital vs. maximum price reached for functions $F_1(x)$, $F_2(x)$, and $F_3(x)$]
Protocol

Initial capital $K_0 := 1$
Initial price $\omega_0 := 1$

For day $t = 1, 2, \ldots$

1. **Investor** takes position $S_t \in \mathbb{R}$
2. **Market** reveals price $\omega_t \in [0, \infty)$
3. Capital becomes $K_t := K_{t-1} + S_t(\omega_t - \omega_{t-1})$

A position $S_t < 0$ is called short
$S_t > 0$ is called long
$S_t > K_{t-1}/\omega_{t-1}$ is called leveraged

Bankrupt when capital $K_t < 0$ is negative.

No assumptions about price-generating process. Full information

Koolen, Vovk (RHUL)
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A position

- \( S_t < 0 \) is called **short**
- \( S_t > 0 \) is called **long**
- \( S_t > K_{t-1}/\omega_{t-1} \) is called **leveraged**

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Bankrupt when capital $K_t < 0$ is negative.

No assumptions about price-generating process. Full information
A strategy prescribes position $S_t$ based on the past prices $\omega_0, \ldots, \omega_{t-1}$.

**Definition**

A function $F : [1, \infty) \to [0, \infty)$ is called an adjuster if there is a strategy that guarantees

$$K_t \geq F \left( \max_{0 \leq s \leq t} \omega_s \right).$$

An adjuster $F$ is admissible if it is not strictly dominated.
Fix a price level $u \geq 1$. The threshold adjuster

$$F_u(y) := u\mathbf{1}_{\{y \geq u\}}$$
Threshold adjusters

Fix a price level \( u \geq 1 \). The threshold adjuster

\[
F_u(y) := u 1_{\{y \geq u\}}
\]

is witnessed by the threshold strategy \( S_u \) that

- takes position 1 until the price first exceeds level \( u \).
- takes position 0 thereafter

Koolen, Vovk (RHUL)
Consider a right-continuous and increasing candidate guarantee $F$.

**Theorem (Characterisation)**

$F$ is an **adjuster** iff

$$\int_{1}^{\infty} \frac{F(y)}{y^2} \, dy \leq 1.$$

Moreover, $F$ is **admissible** iff this holds with equality.
Consider a right-continuous and increasing candidate guarantee $F$.

**Theorem (Characterisation)**

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$$\int_1^\infty \frac{F(y)}{y^2} \, dy \leq 1.$$  

Moreover, $F$ is **admissible** iff this holds with equality.

**Theorem (Representation)**

$F$ is an **adjuster** iff there is a probability measure $P$ on $[1, \infty)$ such that

$$F(y) \leq \int F_u(y) \, dP(u),$$

again with equality iff $F$ is **admissible**.
Fix an adjuster $F$, and consider the strategy that witnesses $F$. If the price $(\omega_t)_{t \geq 0}$ is a martingale, then so is the capital $(K_t)_{t \geq 0}$.

For each $t$, we must have

$$1 = K_0 = \mathbb{E}[K_t] \geq \mathbb{E}[F\left(\max_{0 \leq s \leq t} \omega_s\right)]$$

For many martingales (e.g. Brownian motion), the random variable $\max_{0 \leq s \leq t} \omega_s$ has density $\frac{1}{h^2}$ on $[1, \infty)$, so

$$\mathbb{E}\left[F\left(\max_{0 \leq s \leq t} \omega_s\right)\right] = \int_1^\infty \frac{F(h)}{h^2} \, dh$$
Lower bound flavour

Now take a differentiable $F$ for which $\int_1^\infty F(y)y^{-2}\,dy = 1$.

Idea: write $F$ as a convex combination of threshold guarantees $F_u$.
Now take a differentiable $F$ for which $\int_1^\infty F(y)y^{-2}\,dy = 1$.

Idea: write $F$ as a convex combination of threshold guarantees $F_u$.

Solve for weights $p_{\{1\}}$ and $p(u)$ in

$$F(y) = p_{\{1\}} F_1(y) + \int_1^\infty F_u(y)p(u)\,du = p_{\{1\}} + \int_1^y up(u)\,du$$
Now take a differentiable $F$ for which $\int_{1}^{\infty} F(y)y^{-2} \, dy = 1$.

Idea: write $F$ as a **convex combination of threshold guarantees** $F_u$

Solve for weights $p_{\{1\}}$ and $p(u)$ in

$$F(y) = p_{\{1\}} F_1(y) + \int_{1}^{\infty} F_u(y)p(u) \, du = p_{\{1\}} + \int_{1}^{y} up(u) \, du$$

We get

$$p_{\{1\}} = F(1) \quad \text{and} \quad p(u) = \frac{F'(u)}{u}$$
Now take a differentiable $F$ for which $\int_1^\infty F(y)y^{-2} \, dy = 1$.

Idea: write $F$ as a convex combination of threshold guarantees $F_u$.

Solve for weights $p\{1\}$ and $p(u)$ in

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We get

$$p\{1\} = F(1) \quad \text{and} \quad p(u) = \frac{F'(u)}{u}$$

Now verify that weights are probabilities:

$$p\{1\} + \int_1^\infty p(u) \, du = F(1) + \int_1^\infty \frac{F'(u)}{u} \, du = \int_1^\infty \frac{F(u)}{u^2} \, du = 1$$
What just happened

Sell high:
- We classified candidate guarantees using a simple formula
  - ($\leq 1$) Attainable adjuster
  - ($= 1$) Admissible adjuster
  - ($> 1$) Not an adjuster
- We reverse engineered a strategy for each guarantee
  - Mixture of threshold (sell-at-level) strategies
Outline

1. Introduction
2. Sell high
3. Buy low, sell high
Goal

We start with capital 1$
We consider trading in a single security (with initial price 1$/share)
We want to become rich when the share price exhibits a large upcrossing
([a, b] is upcrossed when the price drops below a before it exceeds b)

A financial expert claims to have a secret strategy that will accomplish our goal. She shows us a function $G$, and guarantees to keep our capital above $G(a, b)$ for all upcrossed intervals $[a, b]$

Ideally, $G(a, b)$ is close to $b / a$

We would like to find out:
Is guaranteeing $G$ possible?
Can more than $G$ be guaranteed?
Can we reverse engineer a strategy for $G$?
Goal

We start with capital 1$
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A financial expert claims to have a secret strategy that will accomplish our goal. She shows us a function $G$, and guarantees to keep our capital above $G(a, b)$ for all upcrossed intervals $[a, b]$. Ideally, $G(a, b)$ is close to $b/a$. We would like to find out:

- Is guaranteeing $G$ possible?
- Can more than $G$ be guaranteed?
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Goal

We start with capital 1$
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Ideally, \( G(a, b) \) is close to \( b/a \).
Goal

We start with capital $1$
We consider trading in a single security (with initial price $1$/share)
We want to become rich when the share price exhibits a large upcrossing ($[a, b]$ is upcrossed when the price drops below $a$ before it exceeds $b$)

A financial expert claims to have a secret strategy that will accomplish our goal. She shows us a function $G$, and guarantees to

keep our capital above $G(a, b)$ for all upcrossed intervals $[a, b]$

Ideally, $G(a, b)$ is close to $b/a$.

We would like to find out:

- Is guaranteeing $G$ possible?
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Definition

A price path \( \omega_0, \ldots, \omega_t \) upcrosses interval \([a, b]\) if

there are \(0 \leq t_a \leq t_b \leq t\) s.t. \(\omega_{t_a} \leq a\) and \(\omega_{t_b} \geq b\).
A price path \( \omega_0, \ldots, \omega_t \) upcrosses interval \([a, b]\) if
\[
\text{there are } 0 \leq t_a \leq t_b \leq t \text{ s.t. } \omega_{t_a} \leq a \text{ and } \omega_{t_b} \geq b.
\]

A strategy prescribes position \( S_t \) based on the past prices \( \omega_0, \ldots, \omega_{t-1} \).

A function \( G : (0, 1] \times [0, \infty) \rightarrow [0, \infty) \) is called an adjuster if there is a strategy that guarantees
\[
K_t \geq G(a, b)
\]
for each \([a, b]\) upcrossed by \( \omega_0, \ldots, \omega_t \).

An adjuster \( G \) is admissible if it is not strictly dominated.
More of the same

Fix price levels $\alpha < \beta$. The threshold adjuster

$$G_{\alpha, \beta}(a, b) = \frac{\beta}{\alpha} \mathbf{1}_{\{a \leq \alpha\}} \mathbf{1}_{\{b \geq \beta\}}$$

is witnessed by the threshold strategy $S_{\alpha, \beta}$ that

- takes position 0 until the price drops below $\alpha$
- takes position $1/\alpha$ until the price rises above $\beta$
- takes position 0 thereafter
More of the same

Fix price levels $\alpha < \beta$. The threshold adjuster

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- takes position 0 until the price drops below $\alpha$
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- takes position 0 thereafter

Optimal strategies allocate their 1$ to threshold strategies according to some probability measure $P(\alpha, \beta)$, and hence achieve

$$G_P(a, b) = \int G_{\alpha, \beta}(a, b) \, dP(\alpha, \beta).$$
More of the same

Fix price levels $\alpha < \beta$. The threshold adjuster

$$G_{\alpha, \beta}(a, b) = \frac{\beta}{\alpha} 1_{\{a \leq \alpha\}} 1_{\{b \geq \beta\}}$$

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Optimal strategies allocate their 1$ to threshold strategies according to some probability measure $P(\alpha, \beta)$, and hence achieve

$$G_P(a, b) = \int G_{\alpha, \beta}(a, b) \, dP(\alpha, \beta).$$

$G_P$ is typically strictly dominated
Mixtures of thresholds are generally dominated

\[ G(a, b) := \frac{1}{2} G_{1, 2}(a, b) + \frac{1}{2} G_{\frac{1}{2}, 1}(a, b) = 1\{a \leq 1 \text{ and } b \geq 2\} + 1\{a \leq \frac{1}{2} \text{ and } b \geq 1\}. \]
Mixtures of thresholds are generally dominated

\[ G(a, b) := \frac{1}{2} G_{1,2}(a, b) + \frac{1}{2} G_{\frac{1}{2},1}(a, b) = \mathbf{1}_{\{a \leq 1 \text{ and } b \geq 2\}} + \mathbf{1}_{\{a \leq \frac{1}{2} \text{ and } b \geq 1\}}. \]
Mixtures of thresholds are generally dominated

\[ G(a, b) := \frac{1}{2} G_{1,2}(a, b) + \frac{1}{2} G_{\frac{1}{2},1}(a, b) = 1_{\{a \leq 1 \text{ and } b \geq 2\}} + 1_{\{a \leq \frac{1}{2} \text{ and } b \geq 1\}}. \]
The GUT of Adjusters

Let $G$ be left/right continuous and de/increasing.

**Theorem (Characterisation)**

$G$ is an adjuster iff

$$
\int_0^\infty 1 - \exp \left( \int_0^1 \frac{1}{a - \inf\{b \mid G(a, b) \geq h}\, da} \right) dh \leq 1.
$$

Moreover, $G$ is admissible iff this holds with equality.

Koolen, Vovk (RHUL)
The GUT of Adjusters

Let $G$ be left/right continuous and de/increasing.

**Theorem (Characterisation)**

$G$ is an **adjuster** iff

$$\int_{0}^{\infty} 1 - \exp \left( \int_{0}^{1} \frac{1}{a - \inf \{ b \mid G(a, b) \geq h \}} \, da \right) \, dh \leq 1.$$  

Moreover, $G$ is **admissible** iff this holds with equality.

**Theorem (Representation)**

$G$ is an **adjuster** iff there are a probability measure $Q$ on $[0, \infty)$ and a nested family $(I_h)_{h \geq 0}$ of north-west sets such that

$$G(a, b) \leq \int G_{I_h}(a, b) \, dQ(h),$$

with equality iff $G$ is **admissible**.
Reverse engineering I

**Definition**

A set \( I \subseteq (0, 1] \times [0, \infty) \) is called **north-west** if \((a, b) \in I\) implies \((0, a] \times [b, \infty) \subseteq I\).

We associate to each north-west set its **frontier**

\[
f_I(a) := \inf\{ b \geq a \mid (a, b) \in I \}.
\]
Fix a north-west set \( I \). The **north-west adjuster**

\[
G_I(a, b) := \frac{1_{\{(a,b)\in I\}}}{1 - \exp \left( \int_0^1 \frac{1}{x - f_i(x)} \, dx \right)} = \frac{1_{\{b \geq f_i(a)\}}}{1 - \exp \left( \int_0^1 \frac{1}{x - f_i(x)} \, dx \right)}.
\]

is witnessed by the **north-west** strategy \( S_I \), which takes position

\[
S_I(\omega_0, \ldots, \omega_{t-1}) = \frac{1_{f_i(m) \geq m}}{f_i(m) - m \exp \left( \int_0^m \frac{1}{a - f_i(a)} \, da \right)} \frac{1 - \exp \left( \int_0^1 \frac{1}{x - f_i(x)} \, dx \right)}{1 - \exp \left( \int_0^1 \frac{1}{x - f_i(x)} \, dx \right)}
\]

where \( m = \min_{0 \leq s < t} \omega_t \)

until \( \omega_t \geq f_i(m) \).

Buys more shares when the global minimum sinks.
Buy low, sell high:

- The intuitive extension fails
  - Mixtures of threshold guarantees are strictly dominated.
  - We need temporal reasoning to appreciate that
- We classified candidate guarantees using a simple formula
  - \((\leq 1)\) Attainable adjuster
  - \((= 1)\) Admissible adjuster
  - \((> 1)\) Not an adjuster
- The formula is not explicitly temporal
- We reverse engineered a strategy for each guarantee
  - Mixture of north-west-set strategies
• Sell high, buy low, then sell high again.
• ...
Thank you!