Adversarial learning of a time interval

Wouter M. Koolen

CWI Centrum Wiskunde & Informatica

UNIVERSITY OF TWENTE.

Symposium in honour of the PhD defence of Robin Markwitz
Friday 26th September, 2025

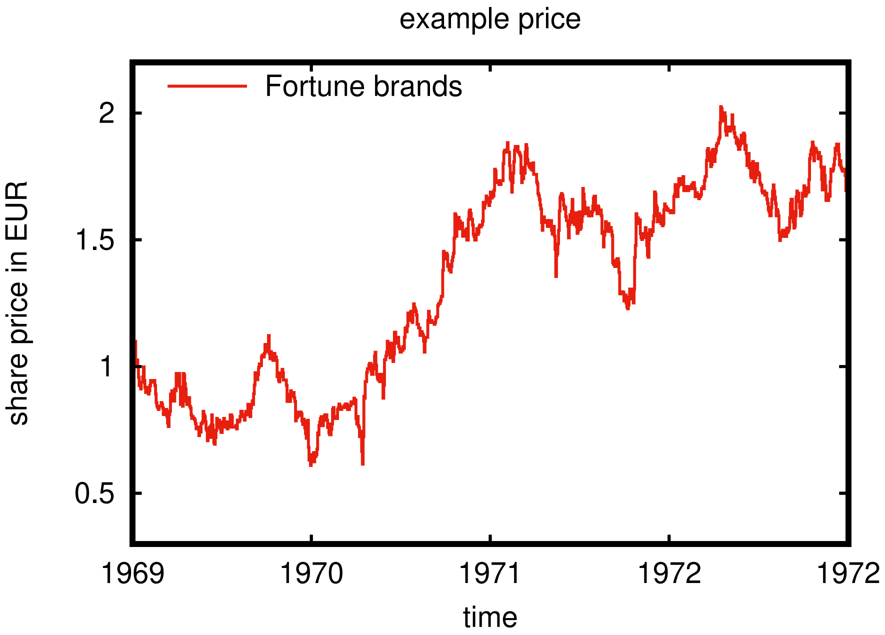
Road map

Context

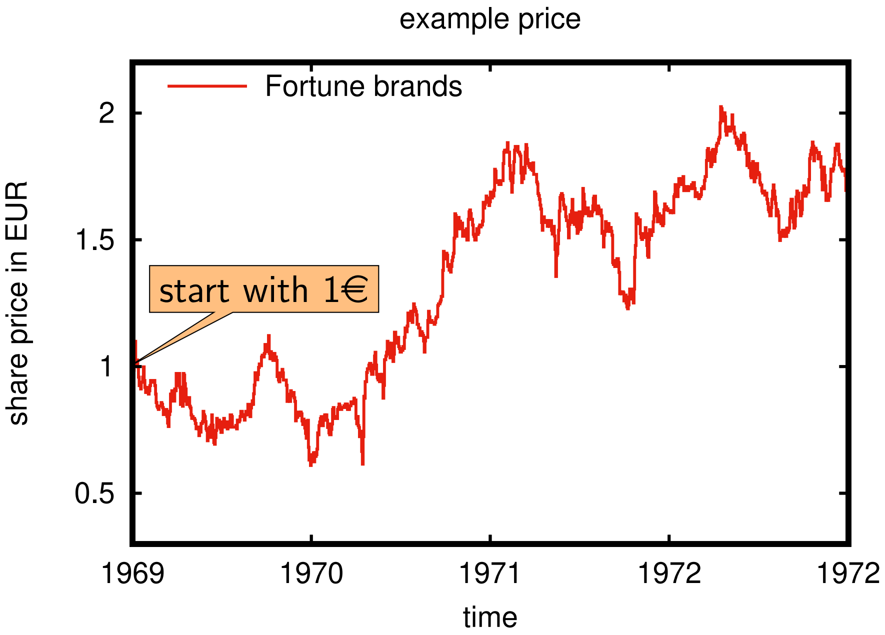
We will be learning a single time interval

- Adversarial data (not stochastic processes)
- Online model (not batch inference)
- Competitive guarantees (not Bayesian)

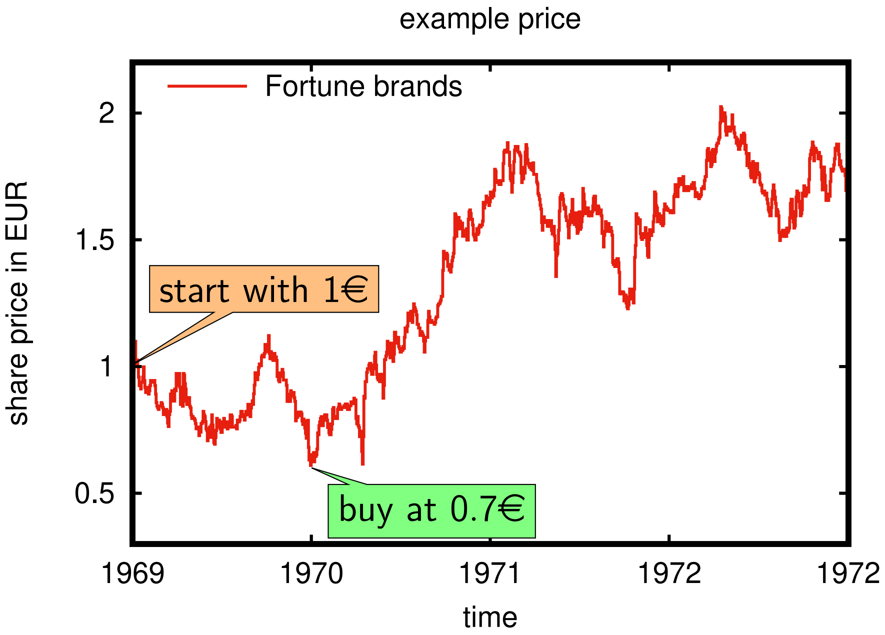
Based on (Koolen and Vovk, 2014)



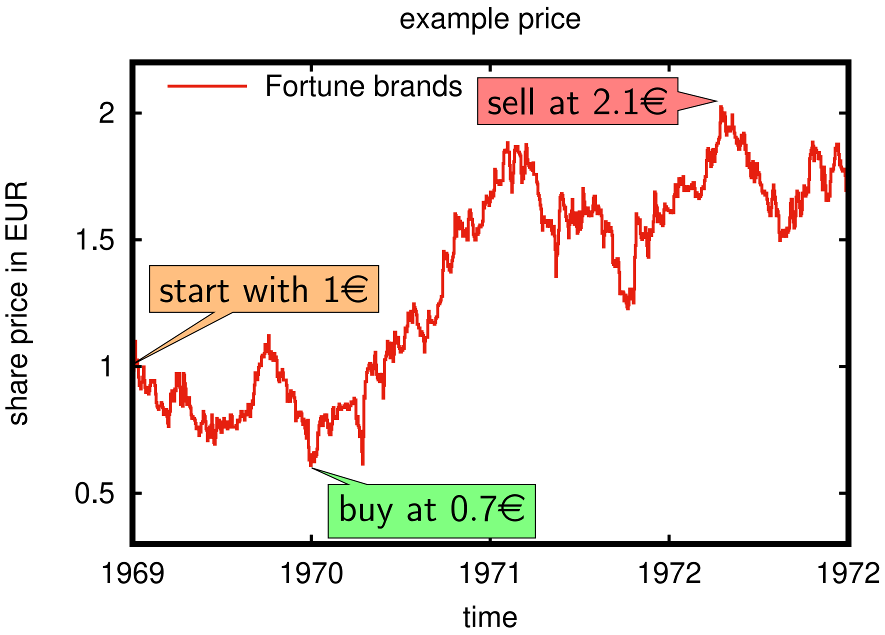
example price
Fortune brands
2
1.5
1
0.5
share price in EUR
1969
1970
1971
1972
1972
time

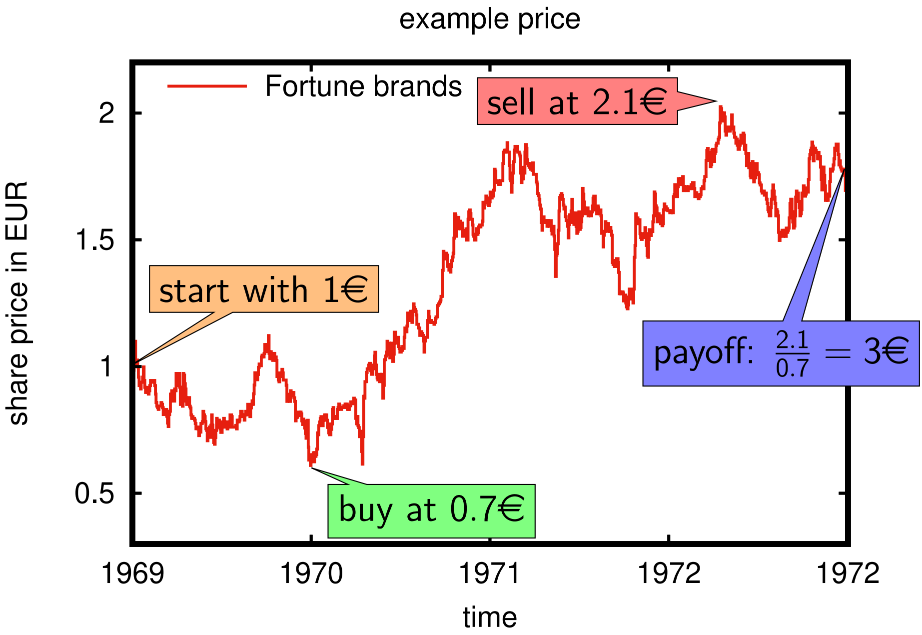


example price
Fortune brands
start with 1€
share price in EUR
2
1.5
1
0.5
1969
1970
1971
1972
1972
time



example price
Fortune brands
share price in EUR
2
1.5
1
0.5
start with 1€
buy at 0.7€
1969
1970
1971
1972
1972
time


example price
Fortune brands
sell at 2.1€
start with 1€
buy at 0.7€
share price in EUR
2
1.5
1
0.5
1969
1970
1971
1972
1972
time



example price
Fortune brands
sell at 2.1€
start with 1€
buy at 0.7€
payoff: $\frac{2.1}{0.7} = 3$€
2
1.5
1
0.5
share price in EUR
1969
1970
1971
1972
1972
time

At a glance

- "Buy low, sell high" a desirable target
- Simple: just need low and high trading price
- Problem: Good trading prices *depend on the data*.

At a glance

- "Buy low, sell high" a desirable target
- Simple: just need low and high trading price
- Problem: Good trading prices *depend on the data*.
- Idea: Could we learn best low/high prices?
- I.e. by online trading guarantee payoff as if we knew them?

At a glance

- "Buy low, sell high" a desirable target
- Simple: just need low and high trading price
- Problem: Good trading prices *depend on the data*.
- Idea: Could we learn best low/high prices?
- I.e. by online trading guarantee payoff as if we knew them?

Answer: a firm and crisp almost

At a glance

- "Buy low, sell high" a desirable target
- Simple: just need low and high trading price
- Problem: Good trading prices *depend on the data*.
- Idea: Could we learn best low/high prices?
- I.e. by online trading guarantee payoff as if we knew them?

Answer: a firm and crisp almost

Our work: complete characterisation of that "almost".

Protocol

Initial capital $K_0 := 1$
Initial price $\omega_0 := 1$

For day $t = 1, 2, \ldots$

1. **Investor** takes position $S_t \in \mathbb{R}$
2. **Market** reveals price $\omega_t \in [0, \infty)$
3. Capital becomes $K_t := K_{t-1} + S_t(\omega_t - \omega_{t-1})$

Protocol

Initial capital $K_0 := 1$
Initial price $\omega_0 := 1$

For day $t = 1, 2, \ldots$

1. **Investor** takes position $S_t \in \mathbb{R}$
2. **Market** reveals price $\omega_t \in [0, \infty)$
3. Capital becomes $K_t := K_{t-1} + S_t(\omega_t - \omega_{t-1})$

A position

- $S_t < 0$ is called short
- $S_t > 0$ is called long
- $S_t > K_{t-1}/\omega_{t-1}$ is called leveraged

Protocol

Initial capital $K_0 := 1$
Initial price $\omega_0 := 1$

For day $t = 1, 2, \ldots$

1. **Investor** takes position $S_t \in \mathbb{R}$
2. **Market** reveals price $\omega_t \in [0, \infty)$
3. Capital becomes $K_t := K_{t-1} + S_t(\omega_t - \omega_{t-1})$

A position

- $S_t < 0$ is called short
- $S_t > 0$ is called long
- $S_t > K_{t-1}/\omega_{t-1}$ is called leveraged

Bankrupt when capital $K_t < 0$ is negative.

Protocol

Initial capital $K_0 := 1$
Initial price $\omega_0 := 1$

For day $t = 1, 2, \ldots$

1. **Investor** takes position $S_t \in \mathbb{R}$
2. **Market** reveals price $\omega_t \in [0, \infty)$
3. Capital becomes $K_t := K_{t-1} + S_t(\omega_t - \omega_{t-1})$

A position

- $S_t < 0$ is called short
- $S_t > 0$ is called long
- $S_t > K_{t-1}/\omega_{t-1}$ is called leveraged

Bankrupt when capital $K_t < 0$ is negative.

No assumptions about price-generating process. Full information

Adjuster

Definition

A price path $\omega_0, \ldots, \omega_t$ upcrosses interval $[a, b]$ if

$$\text{there are } 0 \leq t_a \leq t_b \leq t \quad \text{s.t.} \quad \omega_{t_a} \leq a \text{ and } \omega_{t_b} \geq b.$$

Adjuster

Definition

A price path $\omega_0, \ldots, \omega_t$ upcrosses interval $[a, b]$ if

$$\text{there are } 0 \leq t_a \leq t_b \leq t \quad \text{s.t.} \quad \omega_{t_a} \leq a \text{ and } \omega_{t_b} \geq b.$$

Definition

A strategy prescribes position S_t based on the past prices $\omega_0, \ldots, \omega_{t-1}$.

Adjuster

Definition

A price path $\omega_0, \ldots, \omega_t$ upcrosses interval $[a, b]$ if

$$\text{there are } 0 \leq t_a \leq t_b \leq t \quad \text{s.t.} \quad \omega_{t_a} \leq a \text{ and } \omega_{t_b} \geq b.$$

Definition

A strategy prescribes position S_t based on the past prices $\omega_0, \ldots, \omega_{t-1}$.

Definition

A function $G : (0, 1] \times [0, \infty) \to [0, \infty)$ is called an adjuster if there is a strategy that guarantees

$$K_t \;\geq\; G\,(a, b)$$

for each $[a, b]$ upcrossed by $\omega_0, \ldots, \omega_t$.
An adjuster G is admissible if it is not strictly dominated.

Adjuster

Definition

A price path $\omega_0, \ldots, \omega_t$ upcrosses interval $[a, b]$ if

$$\text{there are } 0 \leq t_a \leq t_b \leq t \quad \text{s.t.} \quad \omega_{t_a} \leq a \text{ and } \omega_{t_b} \geq b.$$

Definition

A strategy prescribes position S_t based on the past prices $\omega_0, \ldots, \omega_{t-1}$.

Definition

A function $G : (0, 1] \times [0, \infty) \to [0, \infty)$ is called an adjuster if there is a strategy that guarantees

$$K_t \;\geq\; G\,(a, b)$$

for each $[a, b]$ upcrossed by $\omega_0, \ldots, \omega_t$.
An adjuster G is admissible if it is not strictly dominated.

Sneak peak: Ideal $G(a, b) = b/a$ is not an adjuster. But we can get close.

Road map

Sequential Threshold strategies

Fix price levels $\alpha < \beta$. The threshold adjuster

$$G_{\alpha,\beta}(a, b) \; = \; \frac{\beta}{\alpha} \mathbf{1}_{\{a \leq \alpha\}} \mathbf{1}_{\{b \geq \beta\}}$$

is witnessed by the threshold strategy $S_{\alpha,\beta}$ that

- takes position 0 until the price drops below α
- takes position $1/\alpha$ until the price rises above β
- takes position 0 thereafter

Sequential Threshold strategies

Fix price levels $\alpha < \beta$. The threshold adjuster

$$G_{\alpha,\beta}(a, b) \;=\; \frac{\beta}{\alpha}\mathbf{1}_{\{a \leq \alpha\}}\mathbf{1}_{\{b \geq \beta\}}$$

is witnessed by the threshold strategy $S_{\alpha,\beta}$ that

- takes position 0 until the price drops below α
- takes position $1/\alpha$ until the price rises above β
- takes position 0 thereafter

Conjecture

Optimal strategies allocate their initial 1€ to threshold strategies according to some probability measure $P(\alpha, \beta)$, and hence achieve

$$G_P(a, b) \;=\; \int G_{\alpha,\beta}(a, b) \, \mathrm{d}P(\alpha, \beta).$$

Sequential Threshold strategies

Fix price levels $\alpha < \beta$. The threshold adjuster

$$G_{\alpha,\beta}(a, b) \;=\; \frac{\beta}{\alpha} \mathbf{1}_{\{a \leq \alpha\}} \mathbf{1}_{\{b \geq \beta\}}$$

is witnessed by the threshold strategy $S_{\alpha,\beta}$ that

- takes position 0 until the price drops below α
- takes position $1/\alpha$ until the price rises above β
- takes position 0 thereafter

Conjecture

Optimal strategies allocate their initial 1€ to threshold strategies according to some probability measure $P(\alpha, \beta)$, and hence achieve

$$G_P(a, b) \;=\; \int G_{\alpha,\beta}(a, b) \, \mathrm{d}P(\alpha, \beta).$$

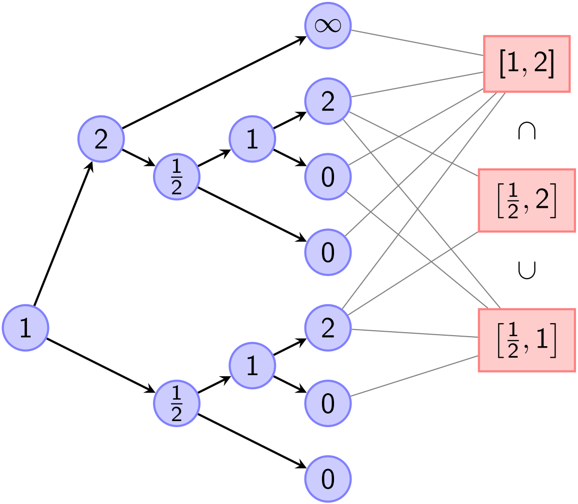
False! G_P is typically strictly dominated

Mixtures of thresholds are generally dominated

$$G(a,b) := \frac{1}{2} G_{1,2}(a,b) + \frac{1}{2} G_{\frac{1}{2},1}(a,b) = \mathbf{1}_{\{a \le 1 \text{ and } b \ge 2\}} + \mathbf{1}_{\left\{a \le \frac{1}{2} \text{ and } b \ge 1\right\}}.$$

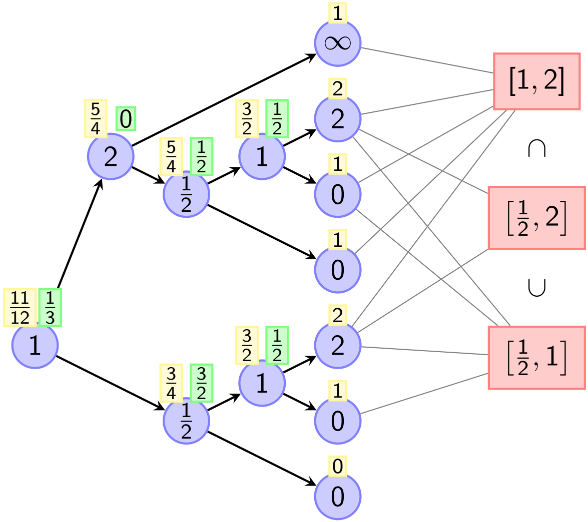
Mixtures of thresholds are generally dominated

$$G(a,b) := \frac{1}{2} G_{1,2}(a,b) + \frac{1}{2} G_{\frac{1}{2},1}(a,b) = \mathbf{1}_{\{a \le 1 \text{ and } b \ge 2\}} + \mathbf{1}_{\left\{a \le \frac{1}{2} \text{ and } b \ge 1\right\}}.$$



Mixtures of thresholds are generally dominated

$$G(a,b) := \frac{1}{2}G_{1,2}(a,b) + \frac{1}{2}G_{\frac{1}{2},1}(a,b) = \mathbf{1}_{\{a \le 1 \text{ and } b \ge 2\}} + \mathbf{1}_{\{a \le \frac{1}{2} \text{ and } b \ge 1\}}.$$



Road map

The GUT of Adjusters

Let G be left/right continuous and de/increasing.

Theorem (Characterisation)

G is an **adjuster** *iff*

$$\int_0^\infty 1 - \exp\left(- \int_{G(a,b) \geq h} \frac{\mathrm{d}a \, \mathrm{d}b}{(b-a)^2} \right) \mathrm{d}h \;\leq\; 1.$$

Moreover, G is **admissible** *iff this holds with equality and G is saturated.*

The GUT of Adjusters

Let G be left/right continuous and de/increasing.

Theorem (Characterisation)

G is an **adjuster** *iff*

$$\int_0^\infty 1 - \exp\left(-\int_{G(a,b)\geq h} \frac{\mathrm{d}a\,\mathrm{d}b}{(b-a)^2}\right) \mathrm{d}h \;\leq\; 1.$$

Moreover, G is **admissible** *iff this holds with equality and G is saturated.*

- Lower bound from option pricing
- Upper bound from explicitly constructed strategy
- Temporal reasoning evaporated.

Road map

Simple adjusters

Corollary (Sell high Dawid, De Rooij, Grünwald, Koolen, Shafer, Shen, Vereshchagin, and Vovk, 2011**)**

Let $G(a, b) := F(b \vee 1)$. G *is an adjuster iff*

$$\int_1^\infty \frac{F(y)}{y^2} \, \mathrm{d}y \;\leq\; 1.$$

Simple adjusters

Corollary (Sell high Dawid, De Rooij, Grünwald, Koolen, Shafer, Shen, Vereshchagin, and Vovk, 2011**)**

Let $G(a, b) := F(b \vee 1)$. *G is an adjuster iff*

$$\int_1^\infty \frac{F(y)}{y^2} \, \mathrm{d}y \;\leq\; 1.$$

Corollary (Length)

Let $G(a, b) := F(b - a)$. *G is an adjuster iff*

$$\int_0^\infty F(y) \frac{e^{-1/y}}{y^2} \, \mathrm{d}y \;\leq\; 1.$$

Simple adjusters

Corollary (Sell high Dawid, De Rooij, Grünwald, Koolen, Shafer, Shen, Vereshchagin, and Vovk, 2011**)**

Let $G(a, b) := F(b \vee 1)$. *G is an adjuster iff*

$$\int_1^\infty \frac{F(y)}{y^2} \, \mathrm{d}y \ \leq \ 1.$$

Corollary (Length)

Let $G(a, b) := F(b - a)$. *G is an adjuster iff*

$$\int_0^\infty F(y) \frac{e^{-1/y}}{y^2} \, \mathrm{d}y \ \leq \ 1.$$

Corollary (Ratio)

Let $G(a, b) := F(b/a)$ *for some unbounded* F. *Then* G *is not an adjuster.*

Our favourite adjuster

Let $0 \leq q < p < 1$. Then

$$G(a, b) \;\coloneqq\; \underbrace{\frac{(b-a)^p}{a^q}}_{\approx b/a} \underbrace{\frac{\left(\frac{p-q}{p}\right)^p}{\Gamma(1-p)}}_{\text{normalisation}}$$

is an admissible adjuster.

Our favourite adjuster

Let $0 \leq q < p < 1$. Then

$$G(a,b) \;:=\; \underbrace{\frac{(b-a)^p}{a^q}}_{\approx b/a} \; \underbrace{\frac{\left(\frac{p-q}{p}\right)^p}{\Gamma(1-p)}}_{\text{normalisation}}$$

is an admissible adjuster.

Strategy: In situation ω with minimum price m take position

$$S(\omega) \;=\; \frac{(p-q)}{m^{1-p+q}} \, \Phi\left(\frac{m^{\frac{p-q}{p}}}{\left(X_G(\omega)\Gamma(1-p)\right)^{1/p}}\right)$$

where Φ is the CDF of the Gamma distribution.

Road map

What just happened

- We took "buy low, sell high" as the learning target
- We consider *parametrised* payoff guarantees
- We classified candidate guarantees using a simple formula
  - (≤ 1) Attainable adjuster
  - ($= 1$) Admissible adjuster
  - (> 1) Not an adjuster
- Looked at some interesting example adjusters

Thank you!

References I

Dawid, A. P., S. De Rooij, P. Grünwald, W. M. Koolen, G. Shafer, A. Shen, N. Vereshchagin, and V. Vovk (Aug. 2011). *Probability-free pricing of adjusted American lookbacks*. Tech. rep. 1108.4113 [q-fin.PR]. arXiv e-prints.

Koolen, W. M. and V. Vovk (Oct. 2014). "Buy low, sell high". In: *Theoretical Computer Science* 558.0. The special issue on Algorithmic Learning Theory for ALT 2012, pp. 144–158.