The t-test is a supermartingale after all

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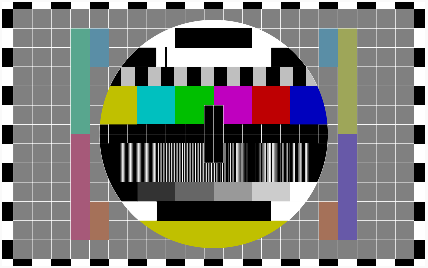
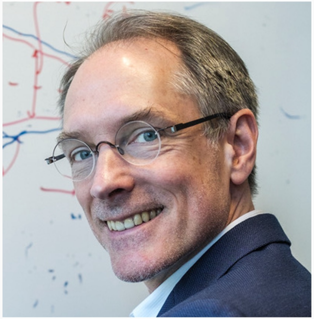
CWI and University of Twente



Warm Thanks



Peter Grünwald

Setup

Simplest composite vs composite example

Consider data stream $X_1, X_2, \ldots$

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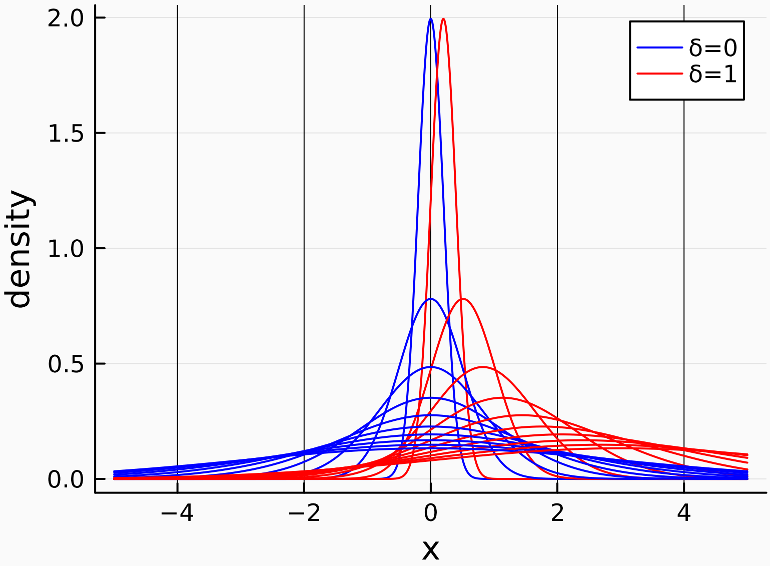
Aim: to disqualify the composite null of **no effect**

$$\mathcal{H}_0 \ = \ \{\delta = 0, \sigma^2 > 0\}$$

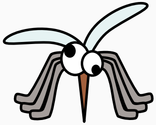
with the help of the composite alternative that the **effect size** is a given $\delta_+ > 0$

$$\mathcal{H}_+ \ = \ \{\delta = \delta_+, \sigma^2 > 0\}$$

Is that hard?



Nuisance



The variance/scale σ^2 is a nuisance parameter.

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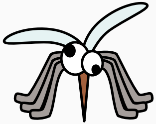

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The nuisance is a **group** (here: positive scaling)

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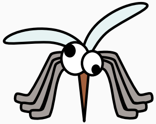

The variance/scale σ^2 is a nuisance parameter.

The nuisance is a **group** (here: positive scaling)

We can quotient it out

- Coarsen the data
- Work with orbits

Coarsening the data

Define the coarsening $(Z_i)_{i\geq 1}$ of the data $(X_i)_{i\geq 1}$ by

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And everyone in $\mathcal{H}_{\delta_+}$ agrees about the distribution of $Z_1, Z_2, \ldots$

yet they don't agree with each other.

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Both null and alternative collapse to a **point**. NB: both see $(Z_i)_{i \geq 1}$ as dependent (not i.i.d.)!

Coarsened Likelihood Ratio

Let $p_0(Z^n)$ and $p_{\delta_+}(Z^n)$ be the density of the coarsening Z^n under the null and the alternative.

Let's look at the process $(M_n)_{n \geq 0}$

$$M_n \;:=\; \frac{p_{\delta_+}(Z^n)}{p_0(Z^n)}$$

Representations of the coarsened likelihood ratio

Let $S_n = \sum_{i=1}^n X_i$ and $V_n = \sum_{i=1}^n X_i^2$ and $R_n = \frac{S_n}{\sqrt{V_n}}$.

We have the Hypergeometric form

$$M_n = \frac{\Gamma\left(\frac{n}{2}\right) {}_1F_1\left(\frac{n}{2}; \frac{1}{2}; \frac{\delta_+^2 R_n^2}{2}\right) + \sqrt{2}\delta_+ R_n \Gamma\left(\frac{n+1}{2}\right) {}_1F_1\left(\frac{n+1}{2}; \frac{3}{2}; \frac{\delta_+^2 R_n^2}{2}\right)}{\Gamma\left(\frac{n}{2}\right) e^{\frac{n}{2}\delta_+^2}}$$

the Pochhammer form

$$M_n = \frac{1}{\Gamma\left(\frac{n}{2}\right) e^{\frac{n}{2}\delta_+^2}} \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{k+n}{2}\right)}{k!} \left(\sqrt{2}\delta_+ R_n\right)^k$$

the Haar forms

$$M_n = \frac{\int p_{\mathcal{N}(\delta_+\sigma, \sigma^2)}(X^n) \frac{1}{\sigma} \, \mathrm{d}\sigma}{\int p_{\mathcal{N}(0, \sigma^2)}(X^n) \frac{1}{\sigma} \, \mathrm{d}\sigma} = \frac{2}{\Gamma\left(\frac{n}{2}\right) e^{\frac{n}{2}\delta_+^2}} \int_0^{\infty} e^{w\sqrt{2}\delta_+ R_n - w^2} w^{n-1} \, \mathrm{d}w$$

the non-central Student-t form

$$M_n = \frac{P\left(T_n; n-1, \delta_+\sqrt{n}\right)}{P\left(T_n; n-1, 0\right)} \qquad \text{where} \qquad T_n = R_n \sqrt{\frac{n-1}{n-R_n^2}}$$

Martingale

Is the coarsened likelihood ratio a martingale for $\mathcal{H}_0$?

$$M_n = \frac{p_{\delta_+}(Z^n)}{p_0(Z^n)}$$

NB: for every $P \in \mathcal{H}_0$

$$\mathbb{E}_P\left[M_{n+1} | Z^n\right] = M_n$$

But

$$\mathbb{E}_P\left[M_{n+1} | X^n\right] \neq M_n$$

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And we stop at the right moment: for every $P \in \mathcal{H}_+$,

$$\mathbb{E}_P\left[\tau\right] \approx \frac{2 \ln \frac{1}{\alpha}}{\ln(1 + \delta_+^2)} = \frac{\ln \frac{1}{\alpha}}{\mathrm{KL}\left(\mathcal{H}_+ \| \mathcal{H}_0\right)}$$

e-variable

Recall our null of zero effect was

$$\mathcal{H}_0 \;=\; \{\delta = 0, \sigma^2 > 0\}.$$

Let's look at the **much larger** null of negative effect

$$\mathcal{H}_{\leq 0} \;:=\; \{\delta \leq 0, \sigma^2 > 0\}$$

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[Pér+24] show that for every $P \in \mathcal{H}_{\leq 0}$ and fixed n,

$$\mathbb{E}_P\left[M_n\right] \;\leq\; 1$$

We say "M_n is an e-variable against $\mathcal{H}_{\leq 0}$".

The Engine of Safety for One-Sided Null

Definition

Random variable T has monotone likelihood ratio (MLR) if, whenever $\delta' \geq \delta$,

$$\frac{p^T_{\delta'}(t)}{p^T_{\delta}(t)} \quad \text{is increasing in } t$$

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Lemma

For MLR variable T, for all $\delta_- \leq 0 \leq \delta_+$,

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In our case, the t-statistic T_n at sample size n has MLR. So $\frac{p_{\delta_+}(T_n)}{p_{\delta_0}(T_n)}$ is an e-variable for $\mathcal{H}_{\leq 0}$.

Summary so far

M_n is

- a **martingale** against every $P \in \mathcal{H}_0$ on the filtration $(\sigma(Z^n))_{n \geq 0}$:

$$\mathbb{E}_{P_0}[M_{n+1} | Z^n] \;=\; M_n$$

- an e-variable for every $P \in \mathcal{H}_{\leq 0}$ on $\sigma(Z^n)$ for every n.

Posing the Problem

Open problem

Recall the null of negative effect

$$\mathcal{H}_{\leq 0} \ := \ \{\delta \leq 0, \sigma^2 > 0\}.$$

Question

Is $(M_n)_{n \geq 0}$ a **supermartingale** against negative effect (i.e. for every $P \in \mathcal{H}_{\leq 0}$)?

Resolution

Monotone Likelihood Ratio

For

$$M_n \;=\; \frac{p_{\delta_+}(Z^n)}{p_0(Z^n)}$$

and $\delta_- \leq 0$, we aim to show

$$\mathbb{E}_{\delta_-}\left[M_{n+1} | Z^n\right] \;\leq\; M_n$$

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Attempt 1: Fix Z^n. Then $\frac{M_{n+1}}{M_n} = \frac{p_{\delta_+}(Z_{n+1}|Z^n)}{p_0(Z_{n+1}|Z^n)}$. Does Z_{n+1} have the monotone likelihood ratio property under the **conditional** model $P_\delta(\cdot|Z^n)$?

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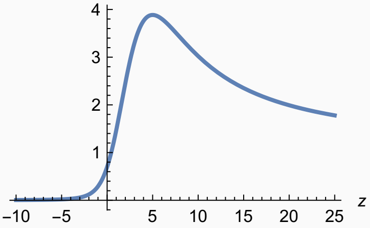


NO!

Sufficiency to the Rescue

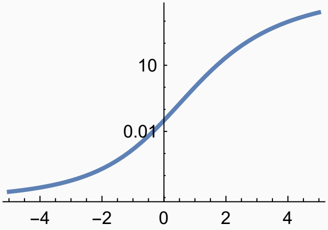
Recall that

$$M_n = \frac{P\left(T_n; n-1, \delta_+\sqrt{n}\right)}{P\left(T_n; n-1, 0\right)}$$

where T_n is the t-statistic. We then have

$$\frac{M_{n+1}}{M_n} = \frac{p_{\delta_+}(Z_{n+1}|Z^n)}{p_0(Z_{n+1}|Z^n)} \overset{\text{sufficiency}}{=} \frac{p_{\delta_+}(T_{n+1}|Z^n)}{p_0(T_{n+1}|Z^n)} \overset{\text{sufficiency}}{=} \frac{p_{\delta_+}(T_{n+1})}{p_0(T_{n+1})}\frac{p_0(Z^n)}{p_{\delta_+}(Z^n)}$$

and so the conditional likelihood ratio is increasing.



YES!

The general case

Theorem

Fix $\delta_0 \leq \delta_+$. *Let* $(T_n)_{n\in\mathbb{N}}$ *be a sequence of* **sufficient statistics** *satisfying the monotone likelihood ratio property. Then the process* $\left(\prod_{i=1}^{n} \frac{p_{\delta_+}^{T_i}(T_i|U^{i-1})}{p_{\delta_0}^{T_i}(T_i|U^{i-1})}\right)_{n\in\mathbb{N}}$ *is identical to the likelihood ratio process* $\left(\frac{p_{\delta_+}(U^n)}{p_{\delta_0}(U^n)}\right)_{n\in\mathbb{N}}$ *and both are "test" (positive, starting at 1)* **supermartingales** *relative to the one-sided null* $\mathcal{H}_{\leq 0}$.

Linear Regression

Linear Regression with Nuisance Covariates

Consider i.i.d. observations (X_i, Y_i, Z_i) from the linear regression model

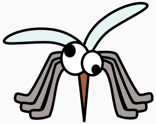
$$Y_i \; = \; \delta\sigma X_i + \beta^\intercal Z_i + \sigma\varepsilon_i,$$

where $\delta \in \mathbb{R}$, $\beta \in \mathbb{R}^d$ and $\sigma \in \mathbb{R}^+$ are the parameters, and $\varepsilon_1, \ldots, \varepsilon_n$ are i.i.d. $\mathcal{N}(0, 1)$.

We aim to test the effect size δ. We treat the **coefficients** β and **scale** σ as nuisance.

$$\mathcal{H}_0 \; = \; \left\{\delta = 0, \sigma^2 > 0, \beta \in \mathbb{R}^d\right\} \qquad \text{vs} \qquad \mathcal{H}_+ \; = \; \left\{\delta = \delta_+, \sigma^2 > 0, \beta \in \mathbb{R}^d\right\}.$$

Result for Linear Regression



In fact, here the nuisance is again a **group** (scaling and general linear).

We can quotient it out, e.g. by coarsening the labels Y^n to

$$U_n := \frac{\boldsymbol{A}_n Y^n}{\|\boldsymbol{A}_n Y^n\|} \in S^{n-d-1} \subseteq \mathbb{R}^{n-d}$$

where $\boldsymbol{A}_n^\intercal \boldsymbol{A}_n Y^n$ is residual of linear regression of labels Y^n onto nuisance covariates $Z_1, \ldots, Z_n$.

Linear Regression ctd

Let us define

$$T_n \ := \ \frac{\frac{b_n^\intercal}{\|b_n\|} U_n}{\frac{1}{\sqrt{n-d-1}} \|\boldsymbol{P}_n U_n\|}$$

where

$$b_n \ := \ \boldsymbol{A}_n X^n \in \mathbb{R}^{n-d} \qquad \text{and} \qquad \boldsymbol{P}_n \ := \ \boldsymbol{I}_n - \frac{b_n b_n^\intercal}{b_n^\intercal b_n}$$

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Then

- T_n is a sufficient statistic for the data $U_1, U_2, \dots, U_n$.
- T_n has non-central Student-t distribution with $n - d - 1$ degrees of freedom and non-centrality parameter $\delta \|b_n\|$
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So $M_n := \frac{f_{\mathrm{T}(n-d-1, \delta_+ \|b_n\|)}(T_n)}{f_{\mathrm{T}(n-d-1,0)}(T_n)}$ is a test supermartingale under the entire null $\mathcal{H}_{\leq 0}$.

Conclusion

- The t-test is a supermartingale after all
- Due the **monotone likelihood ratio** property of a sufficient statistic
- This upgrades to many cases: χ^2, linear regression, ...
- Building block for all sorts of anytime-valid testing and inference

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Let's talk!

References i

[GdHK24] P. D. Grünwald, R. de Heide, and W. M. Koolen. **"Safe Testing".** In: *Journal of the Royal Statistical Society Series B: Statistical Methodology* 86.5 (2024). With Discussion.

[Pér+24] M. F. Pérez-Ortiz, T. Lardy, R. de Heide, and P. Grünwald. **"E-Statistics, Group Invariance and Anytime Valid Testing".** In: *The Annals of Statistics* 52.4 (2024).