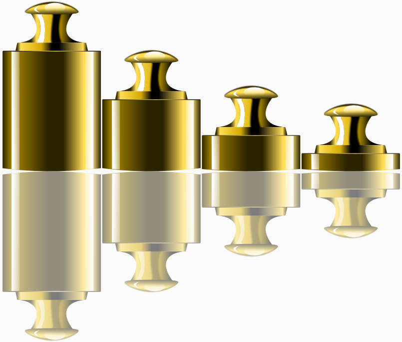
Luckiness in Multi-Scale Online Learning



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KTH

Friday 17th November, 2023

Team effort

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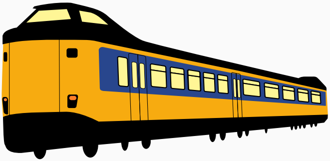
Now: Postdoc at TU Eindhoven

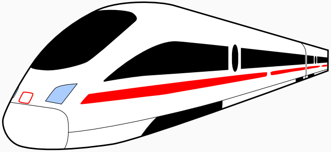
Outline

Motivating Example

Every week I face the choice



$2h \pm 10$ min



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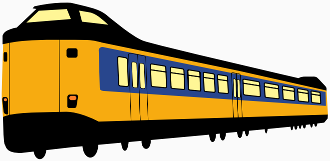

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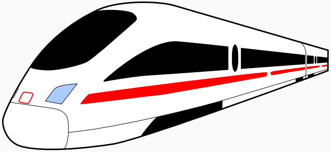

2h $\pm$ 30 min

I want:

- Total travel time $\leq$ best fixed carrier $+$ **small learning overhead**
- By choosing my carrier adaptively (possibly randomised)
- With full information of past service
- Without relying on i.i.d. assumption

Why is this important/interesting

- Fundamental problem with strong connections to
  - (martingale) deviation inequalities
  - convex optimisation and duality
  - (stochastic) gradient descent
  - **uncertainty quantification**
  - bandit problems (partial information)
  - reinforcement learning
  - game theory (saddle point computation)
  - (generic) chaining
  - differential privacy
  - Boosting
  - ...

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  - ...
- Theory well-developed for single loss scale. (Freund and Schapire, 1997; De Rooij et al., 2014; Koolen, Grünwald, and Van Erven, 2016)
- Similar treatment for **multi-scale** was lacking.
  - Existing algorithm templates too rigid
  - No multi-scale Bernstein Inequality

Supervised Learning Theory

What is Statistical Learning

- Receive batch of i.i.d. labelled examples
- Output predictor for new data (e.g. by ERM)
- Prove risk bound using **concentration** (PAC, VC dim)

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- Regret bounded (by minimax rate)
- Risk bounded by **fast rate** if i.i.d. with margin

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A learning problem is Multi-Scale if

- range of losses varies wildly between predictions/actions

Philosophical Note

The learner is uncertain about the overall **best predictor**.

Need to maintain uncertainty. Vague: many implementations.

Online learning provides a crisp framework with a scalar objective.

Hence it informs us about optimal/good/appropriate ways to maintain uncertainty.

The answer is **far from** Bayesian (or perhaps profound generalisation)

Outline

1. Motivation
2. Theory
3. Application

Formal Setup

Fix number K of actions with **known loss ranges** $\boldsymbol{\sigma} \in [0, \infty)^K$

Protocol

for $t = 1, 2, \ldots$

- Learner picks probability distribution $\boldsymbol{w}_t \in \triangle_K$ on actions
- Adversary sets action losses $\boldsymbol{\ell}_t \in \mathbb{R}^K$ with $|\ell_t^k| \leq \sigma_k$
- Learner incurs expected loss $\boldsymbol{w}_t^\intercal \boldsymbol{\ell}_t$

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Definition (Regret)

The **regret** after T rounds with respect to action k is

$$R_T^k \;=\; \sum_{t=1}^T \boldsymbol{w}_t^\intercal \boldsymbol{\ell}_t - \sum_{t=1}^T \ell_t^k$$

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Question

Can Learner keep $R_T^k \leq \sigma_k \sqrt{T}$?

First Step

Consider Follow-the-Regularised-Leader (**FTRL**) template

$$\boldsymbol{w}_t \;=\; \operatorname*{argmin}_{\boldsymbol{w} \in \triangle_K} \; \langle \boldsymbol{w}, \boldsymbol{L}_{t-1} \rangle + D_{\boldsymbol{\eta}}(\boldsymbol{w}, \boldsymbol{u})$$

with **cumulative losses** $\boldsymbol{L}_t = \sum_{s=1}^{t} \boldsymbol{\ell}_s$ and multi-scale entropy

$$D_{\boldsymbol{\eta}}(\boldsymbol{w}, \boldsymbol{u}) \;=\; \sum_k \frac{w_k \ln \frac{w_k}{u_k} - w_k + u_k}{\eta_k}.$$

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Theorem (Bubeck et al., 2019)

FTRL with learning rate $\eta_k = \frac{1}{\sigma_k}\sqrt{\frac{2\ln(K\frac{\sigma_k}{\sigma_{\min}})}{T}}$ *has regret bounded by*

$$R_T^k \;\leq\; \sigma_k \sqrt{T \ln(K \tfrac{\sigma_k}{\sigma_{\min}})}.$$

Matching worst-case regret lower bound.

Are we there yet?

Luckiness?

What if losses $\boldsymbol{\ell}_1, \boldsymbol{\ell}_2, \ldots$ turn out to be i.i.d. after all?

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Luckiness with Margin?

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Strong Contrast

For same-scale case there is a single algorithm with

- Worst-case regret $\sqrt{T \ln K}$ (matching lower bound)
- Stochastic+gap regret $O(1/\Delta)$, a constant(!)
- Interpolates spectrum by data-dependent $\sqrt{V_T \ln K}$ bound.

Main Result

Muscada Algorithm

$$\boldsymbol{w}_t := \operatorname*{argmin}_{\boldsymbol{w} \in \triangle_K} \langle \boldsymbol{w}, \boldsymbol{L}_{t-1} + \boldsymbol{\mu}_{t-1} \rangle + D_{\boldsymbol{\eta}_{t-1}}(\boldsymbol{w}, \boldsymbol{u})$$

where

$$\mu_t^k = \sigma_k \sqrt{v_t \ln(K \tfrac{\sigma_k}{\sigma_{\min}})}$$

$$v_t = 4 \sum_{s=1}^{t} \frac{\operatorname{var}_{\tilde{\boldsymbol{w}}_s}(\boldsymbol{\ell}_s)}{\langle \tilde{\boldsymbol{w}}_s, \boldsymbol{\sigma}^2 \rangle} \qquad \text{with} \qquad \tilde{w}_t^k \propto w_t^k \eta_{t-1}^k$$

$$\eta_t^k = \frac{1}{\sigma_k} \sqrt{\frac{\ln(K \tfrac{\sigma_k}{\sigma_{\min}})}{v_t}} \approx \frac{2}{\sigma_k^2} \frac{\mathrm{d}\mu_t^k}{\mathrm{d}v_t}$$

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Theorem (Main Result)

Muscada guarantees $R_T^k \leq 2\mu_T^k$.

- *Sharpens worst-case regret bound of Bubeck et al. 2019 as $v_t \leq t$.*
- *In i.i.d. setting with gap Δ, expected regret is* **constant** $\mathbb{E}[\mu_T^{k^*}] \leq \frac{K\sigma_{\max}^2}{\Delta}$

Proof: worst-case regret bound

Recall regret $R_t^k = \sum_{s=1}^t \langle \boldsymbol{w}_s, \boldsymbol{\ell}_s \rangle - \sum_{s=1}^t \ell_s^k$. Define **potential function**

$$\Phi_t \;:=\; \Phi(\boldsymbol{R}_t - \boldsymbol{\mu}_t, \boldsymbol{\eta}_t) \;=\; \max_{\boldsymbol{w} \in \triangle_K} \langle \boldsymbol{w}, \boldsymbol{R}_t - \boldsymbol{\mu}_t \rangle - D_{\boldsymbol{\eta}_t}(\boldsymbol{w}, \boldsymbol{u}).$$

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Lemma

Muscada ($\boldsymbol{w}_{t+1} = \arg\max \dots$) *ensures* $0 = \Phi_0 \geq \Phi_1 \geq \dots$.

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Proof.

$$\begin{aligned}
\Phi_t &\leq \Phi(\boldsymbol{R}_t - \boldsymbol{\mu}_t, \boldsymbol{\eta}_{t-1}) && \boldsymbol{\eta} \mapsto D_{\boldsymbol{\eta}} \text{ decr.}\\
&= \Phi(\boldsymbol{R}_t - \boldsymbol{\mu}_{t-1} - \tfrac{1}{2}\boldsymbol{\eta}_{t-1}\boldsymbol{\sigma}^2 \Delta v_t, \boldsymbol{\eta}_{t-1}) && \text{by def. of } \boldsymbol{\mu}_t\\
&= \Phi(\boldsymbol{R}_{t-1} + \langle \boldsymbol{w}_t, \boldsymbol{\mu}_t \rangle - \boldsymbol{\mu}_{t-1}, \boldsymbol{\eta}_{t-1}) && \text{by def. of } \Delta v_t\\
&= \max_{\boldsymbol{w} \in \triangle_K} \langle \boldsymbol{w}, \boldsymbol{R}_{t-1} - \boldsymbol{\mu}_{t-1} \rangle - D_{\boldsymbol{\eta}_{t-1}}(\boldsymbol{w}, \boldsymbol{u}) && \text{by def. of } \Phi\\
&= \langle \boldsymbol{w}_t, \boldsymbol{R}_{t-1} - \boldsymbol{\mu}_{t-1} \rangle - D_{\boldsymbol{\eta}_{t-1}}(\boldsymbol{w}_t, \boldsymbol{u}) && \text{by def. of } \boldsymbol{w}_t\\
&\leq \max_{\boldsymbol{w} \in \triangle_K} \langle \boldsymbol{w}, \boldsymbol{R}_{t-1} - \boldsymbol{\mu}_{t-1} \rangle - D_{\boldsymbol{\eta}_{t-1}}(\boldsymbol{w}, \boldsymbol{u}) && \text{since } \boldsymbol{w}_t \in \triangle_K\\
&= \Phi(\boldsymbol{R}_{t-1} - \boldsymbol{\mu}_{t-1}, \boldsymbol{\eta}_{t-1}) \;=\; \Phi_{t-1} && \text{by def. of } \Phi,\ \Phi_t.
\end{aligned}$$

Proof: worst-case regret bound, ctd

Lemma

The regret compared to expert k is $R_T^k \leq 2\sigma_k \sqrt{v_t \ln(K \frac{\sigma_k}{\sigma_{\min}})}$.

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Proof.

We have $\quad 0 \;\geq\; \Phi_T \;=\; \max_{\boldsymbol{w}\in\triangle_K} \langle \boldsymbol{w}, \boldsymbol{R}_T - \boldsymbol{\mu}_T\rangle - D_{\boldsymbol{\eta}_T}(\boldsymbol{w}, \boldsymbol{u}).$

$$\geq\; \langle \boldsymbol{e}_k, \boldsymbol{R}_T - \boldsymbol{\mu}_T\rangle - D_{\boldsymbol{\eta}_T}(\boldsymbol{e}_k, \boldsymbol{u})$$

Unpacking the divergence, we get

$$R_T^k \;\leq\; \mu_T^k + D_{\boldsymbol{\eta}_T}(\boldsymbol{e}_k, \boldsymbol{u}) \;=\; \mu_T^k + \frac{-\ln u_k - 1}{\eta_T^k} + \sum_j \frac{u_j}{\eta_T^j}$$

$$= \;\sigma_k\sqrt{v_t \ln(K\frac{\sigma_k}{\sigma_{\min}})} + \sigma_k\sqrt{v_t}\frac{-\ln u_k - 1}{\sqrt{\ln(K\frac{\sigma_k}{\sigma_{\min}})}} + \sqrt{v_t}\sum_j \frac{\sigma_j u_j}{\sqrt{\ln(K\frac{\sigma_k}{\sigma_{\min}})}}$$

We pick $u_j = \frac{1}{K}\cdot\frac{\sigma_{\min}}{\sigma_j}$ to get $R_T^k \leq \sqrt{v_t}\left(2\sigma_k\sqrt{\ln(K\frac{\sigma_k}{\sigma_{\min}})} + \sigma_{\min}\right)$ □

Proof: luckiness

Now assume losses are i.i.d. $\boldsymbol{\ell}_t \sim \mathbb{P}$ with gap $\mu(2) - \mu(1) \geq \Delta > 0$.

Lemma (Massart/Bernstein consequence of gap assumption)

Under the gap condition, there is a constant k_{M} such that

$$\mathbf{E}_{\mathbf{P}}[\Delta v_t] \leq k_{\mathrm{M}} \mathbf{E}_{\mathbf{P}}[\boldsymbol{w}_t^\intercal \boldsymbol{\ell}_t - \ell_t^{k^*}],$$

Lemma

The pseudo-regret compared to the best expert $k^* = \operatorname{argmin}_k \mathbb{E}[\ell^k]$ is constant.

Proof.

So then

$$\mathbb{E}[R_T^{k^*}] \;\leq\; \mathbb{E}[\sigma_{k^*} \sqrt{v_T \ldots}] \;\leq\; \sigma_{k^*} \sqrt{\mathbb{E}[v_T] \ldots} \;\leq\; \sigma_{k^*} \sqrt{k_{\mathrm{M}}\, \mathbb{E}[R_T^{k^*}] \ldots}.$$

□

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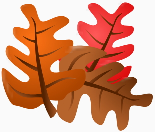
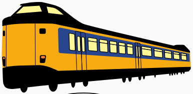
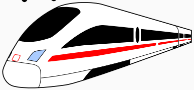
Saddle Point Computation

Now suppose I am paranoid about travel time.

		
	10	90
	0	-10

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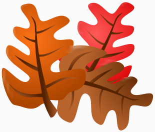
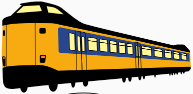
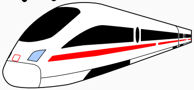
		
	10	90
	0	-10

What is the saddle point? (Hint: it is pure)

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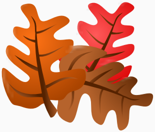
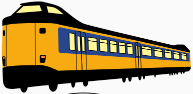
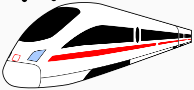
		
	10	90
	0	-10

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Multi-scale:  and  of scale 90 while  and  of scale 10.

In General

Problem

Given (large) payoff matrix M. Compute an ϵ-equilibrium $(\boldsymbol{p}, \boldsymbol{q})$:

$$\max_j \boldsymbol{p}^\intercal M \boldsymbol{e}_j - \min_i \boldsymbol{e}_i^\intercal M \boldsymbol{q} \;\leq\; \epsilon.$$

Popular approach (Freund and Schapire, 1999)

Run online learners $\boldsymbol{p}_t$ and $\boldsymbol{q}_t$ on loss vectors $M\boldsymbol{q}_t$ and $-M^\intercal \boldsymbol{p}_t$.

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Does multi-scale knowledge help?

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Does multi-scale knowledge help?

Yes, sub-optimality gap improves from $\sigma_{\text{max}}/\sqrt{T}$ to $\sigma_{\text{saddle-point}}/\sqrt{T}$.

With optimism (Rakhlin and Sridharan, 2013), empirically $\sigma_{\text{saddle-point}}/T$.

Thanks!

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