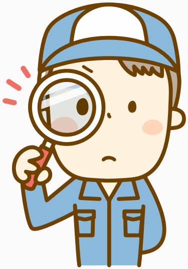
Towards characterizing the first-order query complexity of learning (approximate) Nash equilibria in zero-sum matrix games



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SCOOL seminar
Inria Lille
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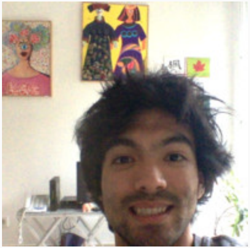
Warm Thanks



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Outline

Games!

Lots of interest, old and new, in solving **convex-concave** min-max problems

$$\min_{p \in \mathcal{P}} \max_{q \in \mathcal{Q}} f(p, q)$$

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Lots of interest, old and new, in solving **convex-concave** min-max problems

$$\min_{p \in \mathcal{P}} \max_{q \in \mathcal{Q}} f(p, q)$$

- Economics
- Optimisation
- Machine learning (GANs)
- Online learning and Bandits (Track-and-Stop)
- ...

What is a solution?

Given $\epsilon \geq 0$, we aim to find an approximate saddle point

$$(p_\star, q_\star) \in \mathcal{P} \times \mathcal{Q},$$

satisfying

$$\max_{q \in \mathcal{Q}} f(p_\star, q) - \min_{p \in \mathcal{P}} f(p, q_\star) \ \leq \ 2\epsilon$$

How are we going to find that solution

We consider the first-order query model.

We start with an unknown f from a known class $\mathcal{F}$.

Interaction protocol

In rounds $1, 2, \ldots, T$

- Learner forwards query (p, q)
- Learner receives **feedback** $(\nabla_p f(p, q), \nabla_q f(p, q))$

The learner outputs an ϵ-optimal saddle point $(p_\star, q_\star)$.

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Query complexity

How many first-order queries are necessary and sufficient for a sequential learner to output an approximate saddle point for any $f \in \mathcal{F}$?

Making the question more tractable

Consider **special case** of zero-sum matrix games (bilinear functions over probability simplex):

$$\mathcal{P} \;=\; \mathcal{Q} \;=\; \triangle_K, \qquad \mathcal{F} \;=\; \left\{ f(p,q) \;=\; p^\intercal M q \;\middle|\; M \in [\pm 1]^{K \times K} \right\}$$

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Algorithms since Brown (1951), up to Rakhlin and Sridharan (2013).

Lower bounds remain elusive.

$\Rightarrow$ Optimal query complexity **unknown**.

What is known: Upper Bounds

1951: First iterative methods by Brown (1951) and Robinson (1951).

1999: Freund and Schapire (1999) discovered the relation to Regret Bounds: Can compute an ϵ-Nash-equilibrium with T iterations, where

$$T = O\left(\frac{\log K}{\epsilon^2}\right)$$

2011: Daskalakis, Deckelbaum, and Kim (2011) can compute an ϵ-Nash-equilibrium with T iterations, where

$$T = O\left(\frac{f(K)}{\epsilon}\right)$$

2013: Rakhlin and Sridharan (2013) can compute an ϵ-Nash-equilibrium with T iterations, where

$$T = O\left(\frac{\log K}{\epsilon}\right)$$

What is known: Lower Bounds

2018: Ouyang and Xu (2021) Showed a lower bound on the query complexity for Saddle-Point Problems with curvature and rotationally invariant constraint sets.

2020: Ibrahim et al. (2020) adapted Nesterov's lower bound technique for games. This requires a two-step linear span assumption.

Our Results

Lower Bounds:

- $K - 2$ queries needed for learning **exact** $\epsilon = 0$ equilibrium
- $\Omega\left(\frac{\log \frac{1}{\epsilon K}}{\log K}\right)$ queries required when $\epsilon \leq \frac{1}{K^4}$.

Upper Bounds:

- If entries in known countable set, say $M \in \mathbb{Q}^{K \times K}$, one query suffices.

Outline

1. Motivation
2. Upper Bounds
3. New Lower Bounds
4. State of the Art
5. Conclusion

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Consider query (p, q) with p arbitrary and $q_j \propto n^{-j}$. Then the i^{th} entry of the feedback (to the p player) is

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Theorem

One query suffices if entries M_{ij} in a known **countable** *set.*

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Why is this **cool**?

Existing lower bound techniques construct matrices with entries from a **finite alphabet**, and hence must be powerless.

- Nesterov-style lower bound (Ibrahim et al., 2020)
- Rademacher entries (Orabona and Pál, 2018)
- Reduction to hard combinatorial / submodular instance (Babichenko, 2016; Hart and Nisan, 2018)

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Ingredients and main Ideas

The fully mixed case is nice. For then $Mq_\star = M^\intercal p_\star \propto \mathbf{1}$.

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If the span of the feedback includes $\mathbf{1}$, then the learner knows an exact equilibrium.

Our job is to sequentially respond to the queries in a consistent way, while keeping $\mathbf{1}$ outside/**far from** the span of the feedback for as long as possible.

Technique

We work with matrices

$$B_0 = \mathcal{B}_{\|\cdot\|_{1,\infty}}\left(\frac{I_K}{2}, \frac{1}{16K^2}\right) = \left\{M \in [\pm 1]^{K\times K} \text{ s.t. } \left|M_{ij} - \frac{\delta_{i=j}}{2}\right| \leq \frac{1}{16K^2}\right\}.$$

Any $M \in B_0$ has

- All equilibria of M are fully mixed
- Non-zero value $\min_p \max_q p^\intercal Mq > 0$.

Main Idea

Consider t rounds with queries

$$(p_s, q_s)_{s \le t}$$

and feedback

$$(\ell_s^{(p)}, \ell_s^{(q)})_{s \le t}$$

Consistent matrices are

$$\mathcal{E}_t \;=\; \left\{ M \in B_0 \,\middle|\, M^\intercal p_s = \ell_s^{(q)} \text{ and } M q_s = \ell_s^{(p)} \text{ for all } s \le t \right\}$$

Key Insight

Lemma

Let (p, q) *be a* **common Nash equilibrium** *for all* $M \in \mathcal{E}_t \neq \emptyset$. *Then* $p \in \mathrm{Span}(p_{1:t})$ *and* $q \in \mathrm{Span}(q_{1:t})$.

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Proof.

Fix $M \in \mathcal{E}_t$. Let $\bar{p} = p - \mathrm{Proj}_{\mathrm{Span}(p_{1:t})}(p)$, and let $u_q \neq \mathbf{0}$ be orthogonal to $\mathrm{Span}(q_{1:t})$. Then $M' = M + \alpha \bar{p} u_q^\intercal \in \mathcal{E}_t$. But then

$$\mathbf{0} \;=\; (M - M')^\intercal p = \alpha(\bar{p}^\intercal p) u_q \;=\; \alpha \|\bar{p}\|^2 u_q$$

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Under same assumption, $\mathbf{1} \in \mathrm{Span}(\ell_{1:t}^{(p)}) \cap \mathrm{Span}(\ell_{1:t}^{(q)})$.

Keeping 1 from the span of the feedback

Theorem

For $T \leq K - 2$ we can sequentially choose $M_t \in \mathcal{E}_t$ s.t. $\mathbf{1} \notin \mathrm{Span}(\ell_{1:T}^{(q)})$.

Keeping 1 from the span of the feedback

Theorem

For $T \le K - 2$ we can sequentially choose $M_t \in \mathcal{E}_t$ s.t. $\mathbf{1} \notin \mathrm{Span}(\ell^{(q)}_{1:T})$.

By induction on t.

For the base case, we pick here $M_0 = 0$ (in paper $M_0 = I_K/2$).

If query in old span $p_{t+1} \in \mathrm{Span}(p_{1:t})$, keep $M_{t+1} = M_t$. Else update

$$M_{t+1} \;=\; M_t + \frac{\bar{p}_{t+1}}{\|\bar{p}_{t+1}\|^2} u_t^\intercal \quad \text{where} \quad \bar{p}_{t+1} \;=\; p_{t+1} - \mathrm{Proj}_{\mathrm{Span}(p_{1:t})}(p_{t+1})$$

and $u_t \neq \mathbf{0}$ is orthogonal to $q_{1:t}$ and to the remainder $\bar{\mathbf{1}}_t = \mathbf{1} - \mathrm{Proj}_{\mathrm{Span}(\ell^{(q)}_{1:t})}(\mathbf{1})$. The feedback is

$$\ell^{(q)}_{t+1} \;=\; M_{t+1} p_{t+1} \;=\; M_t p_{t+1} + u_t \;=\; M_t \bar{p}_{t+1} + M_t(p_{t+1} - \bar{p}_{t+1}) + u_t$$

This is orthogonal to $\bar{\mathbf{1}}_t$. So $\bar{\mathbf{1}}_t$ orthogonal to $\mathrm{Span}(\ell^{(q)}_{1:t+1})$. If $\mathbf{1} \in \mathrm{Span}(\ell^{(q)}_{1:t+1})$, also $\bar{\mathbf{1}}_t \in \mathrm{Span}(\ell^{(q)}_{1:t+1})$ so $\bar{\mathbf{1}}_t = \mathbf{0}$. Contradiction. $\square$

Result

We can keep going until all dimensions are exhausted and we cannot pick u_t orthogonal to $\text{Span}(q_{1:t}, \bar{\mathbf{1}}_t)$. We obtain

Theorem

The query complexity of learning the exact $\epsilon = 0$ nash equilibrium in the first-order query model is $T \geq K - 2$.

Quantitative case

Our result for approximate $\epsilon > 0$ equilibria is based on keeping $\left\| \mathbf{1} - \mathsf{Proj}_{\mathsf{Span}(\ell_{1:t}^{(q)})}(\mathbf{1}) \right\|$ big (instead of non-zero).

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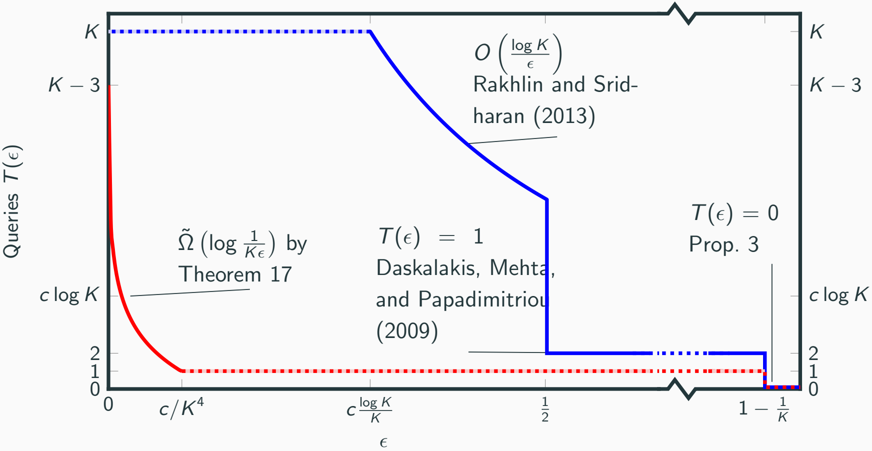
Constant for $\epsilon = \frac{1}{K^c}$. **Insightful** e.g. when $\epsilon = K^{-K}$.

Outline

What we have discussed



Outline

Conclusion

First non-trivial lower bounds.

Lots of open space.

Need even sharper techniques.

Thanks!

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