The Pareto Regret Frontier
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The online learning philosophy

Want to solve a learning problem?

Step 1: Get hold of some experts

Step 2: Aggregate (in production)

Step 3: $$$

The setup

On each round $t$ the learner plays a probability distribution $p_t$ on $K$ experts. Then the vector of expert losses $\ell_t \in [0,1]^K$ is revealed, and the learner suffers

\[ \text{dot loss} = p_t^T \ell_t. \]

After $T$ rounds, the regret w.r.t. expert $k$ is

\[ \text{Regret}^k_T := \sum_{t=1}^T (p_t^T \ell_t - \ell_t^k). \]

A candidate trade-off $r \in \mathbb{R}^K$ is called $T$-realisable if there is a strategy that keeps the regret w.r.t. each $k$ below $r$, i.e., if

\[ \exists p_t : \forall t, \exists \ell_t^k \text{ s.t. } \text{Regret}^k_T \leq r. \]

Normalised large horizon behaviour

\[ \text{limit frontier} := \lim_{T \to \infty} \frac{\text{frontier}_T}{\sqrt{T}}. \]

Asymptotic characterisation ($K = 2$ experts)

The limit frontier is the smooth curve

\[ \langle f(-u), f(+u) \rangle \text{ for } u \in \mathbb{R}, \quad \text{where } f(u) := u\Phi(u) + e^{-\frac{u^2}{2}} \]

and $\Phi(u) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^u e^{-\frac{x^2}{2}} \, dx$. The optimal strategy converges to

\[ p(u) = \Phi(u). \]

Conclusion

Unfairness is a key resource!

Interesting follow-ups:

- $K > 2$ experts exact frontier
- Other losses
- Luckiness stratifications, e.g. $\sqrt{\frac{1}{T}}, \frac{1}{\sqrt{T}}(T-1)$, ...
- Horizon-free biased rates $\rho$: Regret$^k_T \leq \rho T$ for all $t$
- Use of limit algorithm

Problem solved?

HEDGE aggregates $K$ experts s.t. after $T$ rounds

\[ L_T^k - L_T^k \leq \sqrt{\frac{1}{2}T \ln K} \quad \text{for all } k. \]

With tight lower bound

What if

- Lots of experts?
- Special experts?

We need biased aggregation

Simply tune a little differently?

Natural guess: for every distribution $P$ on experts, we can ensure

\[ L_T^k - L_T^k \leq \sqrt{\frac{1}{2}T (-\ln P(k))} \quad \text{for all } k. \]

So what can we guarantee? And how?

Combinatorial characterisation ($K = 2$ experts)

The $T$-realisable Pareto front is piece-wise linear with $T + 1$ vertices:

\[ \langle f_T(i), f_T(T-i) \rangle \quad 0 \leq i \leq T \quad \text{where } f_T(i) := \sum_{j=0}^{i} j^{1/2} T^{(T-j-1)/2}. \]

The optimal strategy at vertex $i$ assigns to expert 1 probability

\[ p_T(0) = 0, \quad p_T(T) = 1, \quad p_T(i) = \frac{f_{T-1}(i) - f_{T-1}(i-1)}{2} \quad 0 < i < T, \]

and it interpolates linearly between neighbouring vertices.

Pareto frontier for $K = 2$ experts. Finite horizons (left). Asymptotic (middle & right)