Learning a set of directions
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Amsterdam 5 meters below sea level

Pump $H_2O$ - but where to point the Windmills?

Online learning to help: for $t = 1, 2, \ldots$
- Mill chooses a randomized direction $u_t \sim P_t$
- Wind reveals direction $x_t$
- Expected gain based on match

Randomized Prediction

For $u \sim P$,
$$\mathbb{E} \left[ (u^T x + c)^2 \right] = x^T \mathbb{E} \left[ uu^T \right] x + 2c x^T \mathbb{E} \left[ u \right] + c^2$$

Key idea: Use parameter $\langle \mu, D \rangle$

What is set $\mathcal{U}$ of valid $\langle \mu, D \rangle$?
$$\mathcal{U} := \{ \langle \mu, D \rangle \mid \exists P : \mu, D \text{ are 1st/2nd moment of some } P \}$$

Characterisation Theorem
Parameter $\langle \mu, D \rangle \in \mathcal{U}$ iff the following semi-definite constraints:
$$\text{tr}(D) = 1 \quad \text{and} \quad D \succeq \mu \mu^T$$

and any $\langle \mu, D \rangle \in \mathcal{U}$ can be efficiently decomposed into $2(n + 1)$ “pure” directions:
$$\langle \mu, D \rangle = \sum_{i=1}^{2(n+1)} w_i \langle u_i, u_i u_i^T \rangle$$

What is a reasonable gain?

**Shifted angle cosine** $\cos \theta := u^T x + 1$

best when $u, x$ parallel, worst when $u, x$ opposite

**Subspace similarity gain** $\cos \theta := (u^T x)^2$

Used in Principle Component Analysis
best when $u, x$ parallel or opposite

**Our directional gain** $\cos \theta := (u^T x + c)^2$

$$= (u^T x)^2 + 2c u^T x + c^2$$

- A tradeoff controlled by mill-dependent constant $c$
- Quadratic Taylor approximation of any gain at $u = 0$

Gradient descent

Mill maintains the two moments $\langle \mu_t, D_t \rangle \in \mathcal{U}$ as parameter
At trial $t = 1 \ldots T$, the Mill

1. Decomposes parameter $\langle \mu_t, D_t \rangle$ into a mixture of directions and draws $u_t$ from mixture
2. Receives Wind direction $x_t$ and gain $\mathbb{E} \left[ (u_t^T x_t + c)^2 \right]$
3. Updates $\langle \mu_t, D_t \rangle$ to $\langle \hat{\mu}_{t+1}, \hat{D}_{t+1} \rangle$ with the gradient of the expected gain on $x_t$
   $$\hat{\mu}_{t+1} := \mu_t + 2c x_t \quad \text{and} \quad \hat{D}_{t+1} := D_t + \eta x_t x_t^T$$
4. Projects $\langle \hat{\mu}_{t+1}, \hat{D}_{t+1} \rangle$ back into $\mathcal{U}$
   $$\langle \mu_{t+1}, D_{t+1} \rangle := \arg\min_{D \succeq \mu \mu^T} \| D - \hat{D}_{t+1} \|^2_F + \| \mu - \hat{\mu}_{t+1} \|^2$$

Theorem

With proper tuning of $\eta$, the regret after $T$ trials of GD is at most $\sqrt{3(4c^2 + 1)T}$
- Regret grows sub-linearly with $T$
- Mill turned close to the best orientation
- Holland is saved 😊

Conclusion

- An efficient method for orienting windmills
- Characterization of set of first two moments of distributions on directions
- Works for $n \geq 2$ dimensions
- We can learn sets of $k \geq 1$ orthogonal directions. Characterisation Theorem and decomposition alg. much more tricky

Bonus plots

Tradeoff constant $c = 1/3$