Combining Adversarial Guarantees and Stochastic Fast Rates in Online Learning

Wouter M. Koolen, Peter Grünwald, Tim van Erven

Goal
We want to make efficient online learning algorithms that adapt automatically to the complexity of the environment.
- Worst-case rates in adversarial environments (safe and robust)
- Fast rates in favorable stochastic environments (practice)

First Step
Consider losses $\ell \subseteq [0,1]$ with stochastic best expert $k^* = \arg \min_u \mathbb{E}[\ell(u)]$ and gap $\min_{k \neq k^*} \mathbb{E}[\ell(u) - \ell(k^*)] > 0$. Then second-order bound (1) implies constant regret $\mathbb{E}[R_T^x] = O(1)$ [Gaillard et al., 2014].

MetaGrad
Modern adaptive algorithms bound regret in terms of variance.

Hinge Loss Example
Unregularized hinge loss on unit ball.
- Data $(x_t, y_t) \sim P$ i.i.d.
- Hinge loss $\ell_t(u) = \max(0, 1 - y_t x_t^\top u)$.
- Mean $\mu = \mathbb{E}[y|x]$ and second moment $D = \mathbb{E}[xx^\top]$.
- Bernstein with $\kappa = 1$ and $B = \frac{2\Delta_{\max}(D)}{\mu}$.

Friendly Stochastic Environments
The Bernstein condition [Bartlett and Mendelson, 2006] says that variance of excess loss is small near stochastic optimum.
Bernstein condition key to fast rates in statistical learning.
Fix $\beta > 0$ and $x \in [0,1]$. We say
- $\ell \sim P$ are $(B, \kappa)$-Bernstein for stochastic experts if
  $$\mathbb{E}[(\ell(u) - \ell(k^*))^2] \leq B(\ell(k^*) - \ell(u))^\kappa$$
  $\forall k, u.$
- $\ell \sim P$ are (linearized) $(B, \kappa)$-Bernstein for stochastic OCO if
  $$\mathbb{E}[(\ell^*(u) - \ell(k^*))^2] \leq B(\ell(k^*) - \ell(u))^\kappa$$
  $\forall (u, k).$
See paper for extensions beyond iid.

Main Theorem
In any stochastic setting satisfying the $(B, \kappa)$-Bernstein condition, a second-order regret bound (1) implies fast rates both in expectation:
$$\mathbb{E}[R_T] = o\left(\sqrt{TV_T} + T^{1/2}\right),$$
and with high probability: for any $\delta > 0$, with probability at least $1 - \delta$,
$$R_T = O\left((K_T - \ln \delta)^{1/2} T^{1/2}\right).$$

Proof Ideas (OCO)
In-expectation for $\kappa = 1$: Consider $\ell \sim P$ with stochastic optimum $u^* = \arg \min_u \mathbb{E}[\ell(u)]$. The second-order regret bound (1) implies
$$\mathbb{E}[R_T] \leq \mathbb{E}[R_T^x] \leq \mathbb{E}\left[\sqrt{V_T} K_T \right] \leq \sqrt{\mathbb{E}[V_T^2] K_T^2}. $$
Let $x_T^u := (u - u^*)^\top \nabla \ell_t(u)$ denote the excess linearized loss of $u$ in round $t$. The Bernstein condition for $\kappa = 1$ yields
$$\mathbb{E}[V_T] = \sum_{t=1}^T \mathbb{E}[(x_T^u)^2] \leq B \sum_{t=1}^T \mathbb{E}[x_T^u] = B \mathbb{E}[R_T],$$
Combining the above two inequalities and solving for $\mathbb{E}[R_T^x]$ gives
$$\mathbb{E}[R_T^x] \leq BK_T^2.$$ For $\kappa < 1$: linearize $(z^2 = x^\top (1 - \kappa) z + x^\top)$ for $z \geq 0$ to show
$$c_1 \cdot e^{1 - x} \mathbb{E}[V_T^x] \leq \mathbb{E}[R_T^x] + c_2 \cdot T \cdot e^{1 - x}.$$ High probability: requires sophisticated martingale argument.