

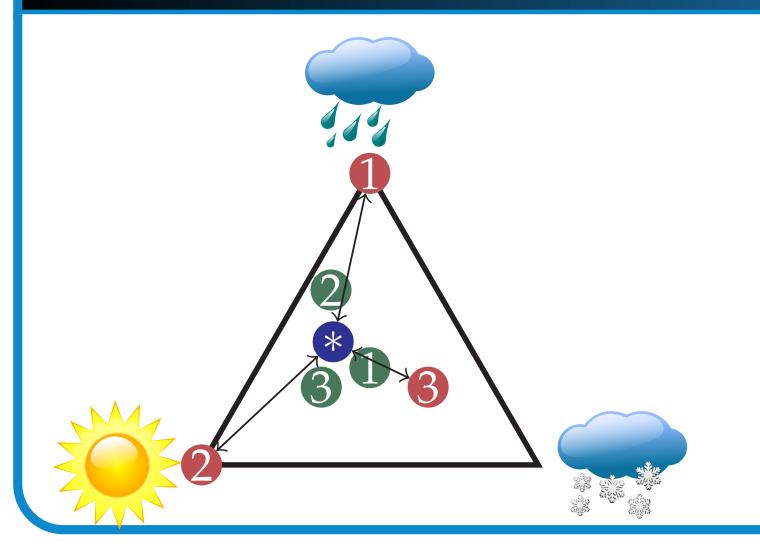
EFFICIENT MINIMAX STRATEGIES FOR SQUARE LOSS GAMES

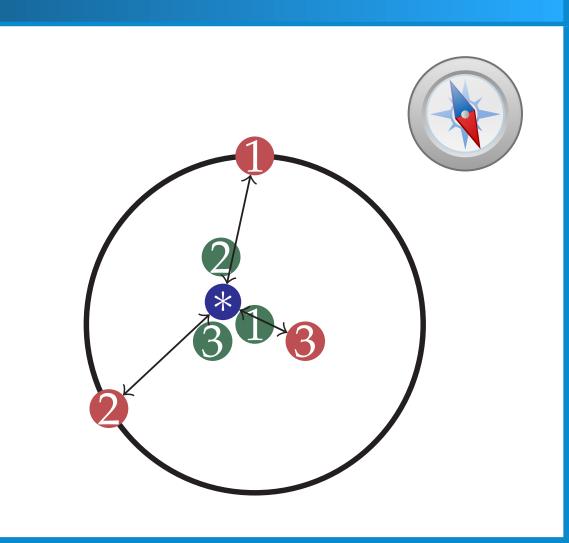
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ONLINE PREDICTION

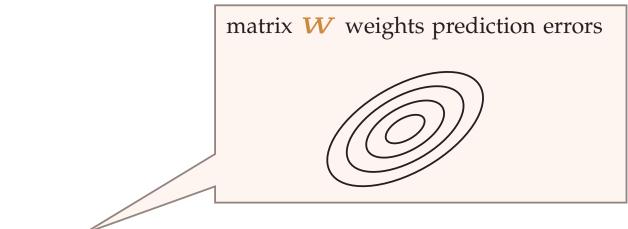




SQUARE LOSS GAMES

Fix a convex set C, positive definite matrix W, length T. For each round t = 1, ..., T,

- We play $a_t \in \mathcal{C}$
- Nature reveals $x_t \in \mathcal{C}$
- We incur loss



$$\ell(\boldsymbol{a}_t, \boldsymbol{x}_t) \coloneqq \|\boldsymbol{a}_t - \boldsymbol{x}_t\|_{\boldsymbol{W}}^2 = (\boldsymbol{a}_t - \boldsymbol{x}_t)^{\intercal} \boldsymbol{W}^{-1} (\boldsymbol{a}_t - \boldsymbol{x}_t)$$

Our goal is to minimize regret w.r.t. best fixed action *a* in hindsight

Regret :=
$$\sum_{t=1}^{T} \ell(\boldsymbol{a}_t, \boldsymbol{x}_t) - \min_{\boldsymbol{a}} \sum_{t=1}^{T} \ell(\boldsymbol{a}, \boldsymbol{x}_t)$$

SOLVING THE GAMES

Using sufficient statistics $\mathbf{s} = \sum_{\tau=1}^{t} \mathbf{x}_{\tau}$ and $\sigma^2 = \sum_{\tau=1}^{t} \mathbf{x}_{\tau}^{\mathsf{T}} \mathbf{W}^{-1} \mathbf{x}_{\tau}$,

Theorem 1 (Brier Game) Let $C = \triangle$. For W satisfying an alignment condition, the value-to-go is

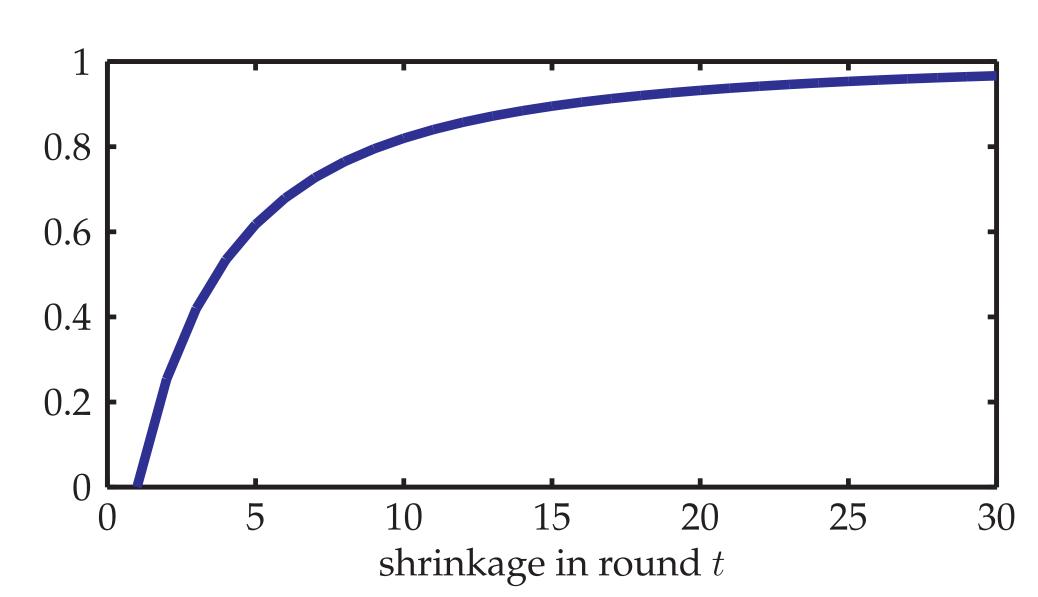
Theorem 2 (Ball Game) Let $C = \bigcirc$. For any positive definite W the value-to-go is

and the minimax and maximin strategies for round t+1 are given by

For round t + 1, the minimax strategy plays

$$egin{aligned} oldsymbol{a}^* &= oldsymbol{p}^* = rac{oldsymbol{s}}{t}tlpha_{t+1} + oldsymbol{c}(1-tlpha_{t+1}) \end{pmatrix} oldsymbol{a}^* &= \left(\lambda_{\max}oldsymbol{I} - (oldsymbol{A}_{t+1} - oldsymbol{W}^{-1})
ight)^{-1}oldsymbol{A}_{t+1}oldsymbol{s} \end{aligned}$$

which is data mean $\frac{s}{t}$ shrunk towards center c.



and the maximin strategy plays two unit length vectors with

$$\Pr\left(oldsymbol{x} = oldsymbol{a}_{\perp} \pm \sqrt{1 - oldsymbol{a}_{\perp}^{\mathsf{T}} oldsymbol{a}_{\perp}} oldsymbol{v}_{ ext{max}}
ight) = rac{1}{2} \pm rac{oldsymbol{a}_{\parallel}^{\mathsf{T}} oldsymbol{v}_{ ext{max}}}{2\sqrt{1 - oldsymbol{a}_{\perp}^{\mathsf{T}} oldsymbol{a}_{\perp}}},$$

where λ_{\max} and v_{\max} correspond to the largest eigenvalue of A_{t+1} and a_{\perp} and a_{\parallel} are the components of a^* perpendicular and parallel to v_{\max} .

All coefficients (α_t , A_t) are efficiently precomputable!

MINIMAX REGRET

If we assume a perfect adversary, how well can we do?

$$V \coloneqq \min_{\boldsymbol{a}_1} \max_{\boldsymbol{x}_1} \dots \min_{\boldsymbol{a}_T} \max_{\boldsymbol{x}_T} \text{Regret}$$

We play to minimize the worst-case regret. We can solve for the $\mathbf{value\text{-}to\text{-}go}\ V$ in any history using the recurrence:

$$V(\boldsymbol{x}_1, \dots, \boldsymbol{x}_T) \coloneqq -\min_{\boldsymbol{a}} \sum_{t=1}^T \ell(\boldsymbol{a}, \boldsymbol{x}_t)$$
 (1)

$$V(\mathbf{x}_1, \dots, \mathbf{x}_{t-1}) := \min_{\mathbf{a}_t} \max_{\mathbf{x}_t} \ell(\mathbf{a}_t, \mathbf{x}_t) + V(\mathbf{x}_1, \dots, \mathbf{x}_t)$$
 (2)

The minimax regret V equals value-to-go $V(\epsilon)$ from empty history. To play the minimax strategy: after seeing $\mathbf{x}_1, \dots, \mathbf{x}_{t-1}$,

- Compute $V(\mathbf{x}_1, \dots, \mathbf{x}_t)$
- Choose a_t as the minimizer of Equation (2)

Problem: this is expensive. Are there examples where V is a simple function of some statistics of $\mathbf{x}_1, \dots, \mathbf{x}_t$ that is simple to *precompute*?

REGRET BOUNDS

- Regret_{Brier} $\propto \sum_{t=1}^{T} \alpha_t$.
- Regret_{Ball} = $\lambda_{\max}(\mathbf{W}^{-1}) \sum_{t=1}^{T} \alpha_t$.
- [1] show that $\sum_{t=1}^{T} \alpha_t = O(\log(T) \log\log(T))$.
- Compare with $O(\log(T))$ of Follow the Leader.

[1] E. Takimoto, M. Warmuth The minimax strategy for Gaussian density estimation In COLT '00

COEFFICIENT RECURSIONS

Brier Game: $\alpha_T = \frac{1}{T}$ and

$$\alpha_t = \alpha_{t+1}^2 + \alpha_{t+1}.$$

Ball Game: $\mathbf{A}_T = \frac{1}{T}\mathbf{W}^{-1}$ and

$$\mathbf{A}_{t} = \mathbf{A}_{t+1} (\mathbf{W}^{-1} + \lambda_{\max} \mathbf{I} - \mathbf{A}_{t+1})^{-1} \mathbf{A}_{t+1} + \mathbf{A}_{t+1},$$

which maintains the eigenvectors of \boldsymbol{W} and updates the eigenvalues.

CONCLUSION

- Both games have tractable value functions and optimal strategies.
- All coefficients can be precomputed (need to know horizon T).
- Our strategies are subgame perfect: we are minimax optimal from every history.
- FOLLOW-UP We now know the minimax strategy for arbitrary outcome space C.