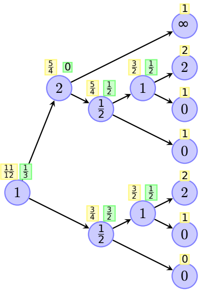
Game-theoretic Probability



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ILLC Project, June 2016

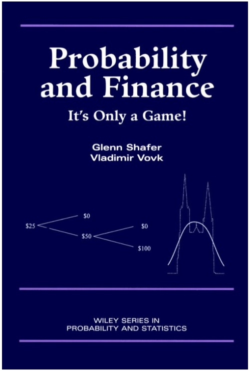
Bio

04–06	MSc	INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION
06–11	PhD	CWI Centrum Wiskunde & Informatica
11–13	Postdoc	Royal Holloway University of London
13–15	Postdoc	QUT Queensland University of Technology and Berkeley UNIVERSITY OF CALIFORNIA
15–	VENI	CWI Centrum Wiskunde & Informatica

Central topic: **learning**

Game-theoretic probability

- Foundation for probability theory based on games (2 player, full information, infinite, ...).
- Alternative to measure theory
  New ingredient: strategies
- Different philosophy
- Different generalization (from MT core)
- Relation with other areas
  - hypothesis testing
  - algorithmic randomness
  - imprecise probability
  - on-line prediction
  - finance
- Applications in machine learning



Example

Infinite sequence of fair coin flips:

$$x_n \in \{0, 1\}$$

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What can we say about empirical frequency

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i \, ?$$

Measure Theory

Construct **measure** $\mathbb{P}$ (and σ-algebra) modeling fair coin. Show

$$\mathbb{P}\left\{\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}x_i = \frac{1}{2}\right\} \;=\; 1$$

"Strong law of large numbers"

Game-theoretic Probability

1. Protocol.
   - $\mathcal{K}_0 = 1$.
   - FOR $n = 1, 2, \ldots$
     - Skeptic announces $M_n \in \mathbb{R}$.
     - Reality announces $x_n \in \{0, 1\}$.
     - $\mathcal{K}_n = \mathcal{K}_{n-1} + M_n \left(x_n - \frac{1}{2}\right)$.

Game-theoretic Probability

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updated capital

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2. Winning condition:

Skeptic **wins** if $\forall n : \mathcal{K}_n \geq 0$ and

either $\displaystyle \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i = \frac{1}{2}$ or $\displaystyle \lim_{n\to\infty} \mathcal{K}_n = \infty$

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3. **Result:** Skeptic has a **winning strategy**.
   (explicit, computable, intuitive, ...)

What gets me ticking

- Broader understanding of reasoning under uncertainty
- Strategies make **learning** operational
  - Success in machine learning
- Inverse perspective: "Defensive forecasting"

Course Organization

first 2 weeks:

- 6 lectures
  - Cover book chapters 1–8
  - Applications to on-line prediction
- homework ($6 \times 10\%$)

second 2 weeks:

- individual research
- progress meeting
- report (40%)

Lectures

- Historical perspective
  Kolmogorov axioms, martingales
  Binomial pricing
- Strong laws (large numbers, iterated logarithm)
- Weak laws (large nubers, central limit)
- Comparing game- and measure-theoretic probability
- Online prediction
  Defensive forecasting

Project Report

The written report should provide either

- some (small) original result or
- a survey of some of the literature with a new perspective

See course website for inspiration and pointers to literature.

What gets you ticking

- Like games. (G is the 5$^{\text{th}}$ letter in ILLC)
- Like foundations.
- Like maths challenge.

See you in June!