Game Theoretic Probability Homework 5

Wednesday 8^{th} June, 2016

This set is due on Friday 10^{th} June, 2016 before class. A definition summary can be found in `http://wouterkoolen.info/GTP_ILLC_2016/notation.pdf`.

Exercise 1:

Fix a measurable space $(\Omega, \mathcal{F})$, and let x and y be measurable functions from Ω to $\mathbb{R}$.

1. Show that αx is measurable for all $\alpha \in \mathbb{R}$.
2. Show that $x + y$ is measurable.
3. Show that xy is measurable.
4. Show that $\max\{x, y\}$ is measurable.

Exercise 2:

Consider the bounded forecasting protocol with sample space $[-1, +1]^\infty$:

> $\mathcal{K}_0 = 1$.
> **for** $n = 1, 2, \ldots$ **do**
>   Forecaster announces $m_n \in [-1, +1]$.
>   Skeptic announces $M_n \in \mathbb{R}$.
>   Reality announces $x_n \in [-1, +1]$.
> $\mathcal{K}_n = \mathcal{K}_{n-1} + M_n(x_n - m_n)$
> **end for**

Let $\mathcal{P}$ be a Borel measurable strategy for Skeptic forcing

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} (x_i - m_i) = 0.$$

We write $\mathcal{P}_n$ for the number of tickets bought by $\mathcal{P}$ in round n.

Now consider using $\mathcal{P}$ in the measure-theoretic probability framework. Let x_1, x_2 be a sequence of $[-1, +1]$-bounded random variables adapted to some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n=1}^\infty, \mathbb{P})$. Show that the process

$$\mathcal{L}_0 = 1 \qquad \text{and} \qquad \mathcal{L}_n = \mathcal{L}_{n-1} + \mathcal{P}_n \left(x_n - \mathbb{E}[x_n | \mathcal{F}_{n-1}]\right)$$

is a measure-theoretic martingale.

Exercise 3:

Remember that the *upper price* of a variable x is defined as

$$\overline{\mathbb{E}}[x] \;=\; \inf\left\{\mathcal{S}_0 | \mathcal{S} \text{ is a capital process and } \liminf \mathcal{S} \geq x\right\}.$$

Upper price is "the smallest price at which Skeptic can buy x" (replicate the payoff x by trading).

Here are two candidate definitions of *lower price*:

$$\underline{\mathbb{E}}[x] \;=\; \sup\left\{-\mathcal{S}_0 | \mathcal{S} \text{ is a capital process and } \liminf \mathcal{S} \geq -x\right\} \qquad (1)$$

$$\underline{\mathbb{E}}[x] \;=\; \sup\left\{\mathcal{S}_0 | \mathcal{S} \text{ is a capital process and } \limsup \mathcal{S} \leq x\right\} \qquad (2)$$

1. Show that in a symmetric protocol these are equivalent.
2. In an asymmetric protocol, which one correctly formalizes "the largest price for which Skeptic can sell x"? Why?
3. For which one do we have $\overline{\mathbb{E}}[x] = -\underline{\mathbb{E}}[-x]$?