Game Theoretic Probability Homework 4

Monday 6^{th} June, 2016

This set is due on Wednesday 8^{th} June, 2016 before class. A definition summary can be found in `http://wouterkoolen.info/GTP_ILLC_2016/notation.pdf`.

Exercise 1:

Consider the Finite Horizon Bounded Forecasting Game with protocol

> Parameters N, $\alpha > 0$, $\epsilon > 0$
> $\mathcal{K}_0 = \alpha$.
> **for** $n = 1, \ldots, N$ **do**
> Skeptic announces $M_n \in \mathbb{R}$.
> Reality announces $x_n \in [-1, +1]$.
> $\mathcal{K}_n = \mathcal{K}_{n-1} + M_n x_n$
> **end for**

Skeptic wins if $\mathcal{K}_n$ is never negative and either $\mathcal{K}_N \geq 1$ or $\left|\frac{1}{N}\sum_{i=1}^N x_i\right| < \epsilon$.

Show that Skeptic has a winning strategy if $N \geq \frac{1}{\alpha\epsilon^2}$.

Exercise 2:

Let

$$\overline{U}(s, D) \;=\; \frac{1}{\sqrt{2\pi D}} \int_{-\infty}^{\infty} U(z) e^{-\frac{(z-s)^2}{2D}} \,\mathrm{d}z.$$

Assume that U satisfies Leibniz's integration rule (i.e. that differentiation may be exchanged with the integral in $\overline{U}$). Show that $\overline{U}$ satisfies the Heat Equation, i.e. that

$$\frac{\partial \overline{U}}{\partial D} \;=\; \frac{1}{2}\frac{\partial^2 \overline{U}}{\partial s^2}$$

Exercise 3:

1. Consider first a one-round game with outcome $x \in \{-\frac{1}{\sqrt{N}}, \frac{1}{\sqrt{N}}\}$, where Skeptic can take any position $M \in \mathbb{R}$ with payoff Mx. Consider the variable $U(x)$, defined by its two values $U(-\frac{1}{\sqrt{N}})$ and $U(\frac{1}{\sqrt{N}})$. Compute $\overline{\mathbb{E}}[U(x)]$ and $\underline{\mathbb{E}}[U(x)]$.

2. Consider the Finite Horizon Fair Coin Forecasting Game with protocol

   Parameters N, $\alpha > 0$, $\epsilon > 0$
   $\mathcal{K}_0 = \alpha$.
   for $n = 1, \ldots, N$ **do**
   Skeptic announces $M_n \in \mathbb{R}$.
   Reality announces $x_n \in \left\{-\frac{1}{\sqrt{N}}, +\frac{1}{\sqrt{N}}\right\}$.
   $\mathcal{K}_n = \mathcal{K}_{n-1} + M_n x_n$
   end for

   Let $\mathcal{S}_n = \sum_{i=1}^n x_i$.
   Use your answer to item 1 recursively to show that

$$\mathbb{E}[U(\mathcal{S}_N)] = \sum_{i=0}^{N} 2^{-N} \binom{N}{i} U\left(\frac{i - N/2}{\sqrt{N/4}}\right).$$

Exercise 4:

Consider the Finite Horizon Bounded Forecasting Game (with fixed mean $m_n = 0$ and variance $v_n = \frac{1}{N}$) with protocol

Parameters N, $\alpha > 0$, $\epsilon > 0$
$\mathcal{K}_0 = \alpha$.
for $n = 1, \ldots, N$ **do**
Skeptic announces $M_n \in \mathbb{R}$, $V_n \in \mathbb{R}$.
Reality announces $x_n \in \left[-\frac{1}{\sqrt{N}}, +\frac{1}{\sqrt{N}}\right]$.
$\mathcal{K}_n = \mathcal{K}_{n-1} + M_n x_n + V_n \left(x_n^2 - \frac{1}{N}\right)$
end for

Note that Skeptic can gamble on small variance by choosing $V_n < 0$. Let $\mathcal{S}_n = \sum_{i=1}^n x_i$.

Let $U : \mathbb{R} \to \mathbb{R}$ be a smooth function constant outside a finite interval. Show that for large enough N, the upper price $\overline{\mathbb{E}}[U(\mathcal{S}_N)]$ and lower price $\underline{\mathbb{E}}[U(\mathcal{S}_N)]$ are both arbitrarily close to

$$\int_{-\infty}^{\infty} U(z) \mathcal{N}_{0,1}(\mathrm{d}z).$$