Game Theoretic Probability Homework 3

Friday 3rd June, 2016

This set is due on Monday 6th June, 2016 before class. A definition summary can be found in `http://wouterkoolen.info/GTP_ILLC_2016/notation.pdf`.

1 Chapter 5

Exercise 1:

Fix a constant $C > 0$. This question is about simulating tickets paying x^2 using just tickets paying x. Consider the zero-mean bounded one-round protocol where Skeptic chooses M, Reality reveals $x \in [-C, C]$ and Skeptic's capital increases by Mx. Compute the upper price $\overline{\mathbb{E}}[x^2]$.

The following questions take place in the Unbounded Forecasting Game (with means set to zero $m_n = 0$), which has protocol

$\mathcal{K}_0 = 1$.
for $n = 1, 2, \ldots$ **do**
  Forecaster announces $v_n \geq 0$.
  Skeptic announces $M_n \in \mathbb{R}$ and $V_n \geq 0$.
  Reality announces $x_n \in \mathbb{R}$.
  $\mathcal{K}_n = \mathcal{K}_{n-1} + M_n x_n + V_n \big(x_n^2 - v_n\big)$
end for

We abbreviate $A_n = \sum_{i=1}^n v_i$ and write $A_n \to \infty$ for $\lim_{n\to\infty} A_n = \infty$.

Exercise 2:

The Law of the Iterated Logarithm states that Skeptic can force

$$\left(A_n \to \infty \text{ and } |x_n| = o\left(\sqrt{\frac{\ln \ln A_n}{A_n}}\right)\right) \quad \Rightarrow \quad \limsup_{n\to\infty} \frac{\sum_{i=1}^n x_i}{\sqrt{2 A_n \ln \ln A_n}} \leq 1.$$

1. Deduce from the above using symmetry of the protocol that Skeptic can force

$$\left(A_n \to \infty \text{ and } |x_n| = o\left(\sqrt{\frac{\ln \ln A_n}{A_n}}\right)\right) \quad \Rightarrow \quad \liminf_{n\to\infty} \frac{\sum_{i=1}^n x_i}{\sqrt{2 A_n \ln \ln A_n}} \geq -1.$$

2. We say that Forecaster and Reality together can *force* an event E if they have a strategy assuring that either Skeptic goes bankrupt ($\mathcal{K}_n < 0$ for some n) or E happens while Skeptic does not become infinitely rich.

   Show that Forecaster and Reality can force

$$A_n \to \infty \text{ and } |x_n| = o\left(\sqrt{\frac{\ln\ln A_n}{A_n}}\right) \text{ and } \lim_{n\to\infty} \frac{\sum_{i=1}^n x_i}{\sqrt{2A_n \ln\ln A_n}} = 0.$$

   Deduce that Skeptic cannot force either of the above inequalities with equality.

Exercise 3:
Fix $\delta > 0$ and $\beta > 0$. Consider the process $\mathcal{K}$ defined by

$$\mathcal{K}_0 \;=\; 1 \qquad \text{and} \qquad \mathcal{K}_n \;=\; \mathcal{K}_{n-1} \frac{1 + \beta x_n + (1+\delta)\frac{\beta^2}{2} x_n^2}{1 + (1+\delta)\frac{\beta^2}{2} v_n}.$$

1. Show that $\mathcal{K}$ is a capital process by finding a strategy $\mathcal{P} = (\mathcal{M}, \mathcal{V})$ for it.
2. Show that $\mathcal{K}$ is non-negative.
3. Show that $\mathcal{P}$ forces

$$\left(\beta x_i \leq \ln(1 + \beta x_i + (1+\delta)\frac{\beta^2}{2} x_i^2) \text{ for all } i\right) \;\Rightarrow$$
$$\exists C : \left(\sum_{i=1}^n x_i \;<\; C/\beta + (1+\delta)\frac{\beta}{2} A_n \text{ for all } n\right). \quad (1)$$

4. Fix A_n and C. Compute the minimum and minimizer of

$$C/\beta + (1+\delta)\frac{\beta}{2} A_n$$

   as a function of β.

5. Your answer to item 4 should be better than $\sqrt{2A_n \ln\ln A_n}$ for large n. How is this possible?