Game Theoretic Probability Homework 2

Wednesday 1^{st} June, 2016

This set is due on Friday 3^{rd} June, 2016 before class. A definition summary can be found in `http://wouterkoolen.info/GTP_ILLC_2016/notation.pdf`.

1 Chapter 3

Exercise 1:

Consider the Fair Coin Game with protocol

> $\mathcal{K}_0 = 1$.
> **for** $n = 1, 2, \dots$ **do**
> Skeptic announces $M_n \in \mathbb{R}$.
> Reality announces $x_n \in \{-1, +1\}$.
> $\mathcal{K}_n = \mathcal{K}_{n-1} + M_n x_n$
> **end for**

Let $s_n = \sum_{i=1}^n x_i$.

1. Show that the strategy

$$M_n = \frac{\mathcal{K}_{n-1} s_{n-1}}{n+1}$$

starting from $\mathcal{K}_0 = 1$ has capital process

$$\mathcal{K}_n = 2^n \frac{\frac{n+s_n}{2}!\,\frac{n-s_n}{2}!}{(n+1)!}.$$

2. The game-theoretic law of large numbers for the Fair Coin Game states that Skeptic has a strategy for guaranteeing $\mathcal{K}_n \geq 0$ and

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n x_i}{n} = 0 \qquad \text{or} \qquad \liminf_{n\to\infty} \mathcal{K}_n = \infty$$

Deduce the game-theoretic law of large numbers from item 1. You may use that $\binom{n}{k} \leq e^{nH(k/n)}$ where $H(p) = -p \ln p - (1-p)\ln(1-p)$ is the binary entropy function (which is a concave continuous function defined on $[0, 1]$ that is maximized at $H(1/2) = \ln 2$).

Exercise 2:
Consider the Bounded Forecasting Game with protocol

> $\mathcal{K}_0 = 1.$
> **for** $n = 1, 2, \ldots$ **do**
>   Skeptic announces $M_n \in \mathbb{R}$.
>   Reality announces $x_n \in [-1, +1]$.
> $\mathcal{K}_n = \mathcal{K}_{n-1} + M_n x_n$
> **end for**

Adapt the proof of Lemma 3.3 to show that Skeptic has a strategy guaranteeing $\mathcal{K}_n \geq 0$ and

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n x_i}{n} = 0 \qquad \text{or} \qquad \limsup_{n\to\infty} \frac{\ln \mathcal{K}_n}{n} > 0$$

i.e. the average converges to zero or Skeptic's capital grows exponentially fast.

Exercise 3:
Consider the Lower Bounded Upper Forecasting Game with protocol

> $\mathcal{K}_0 = 1.$
> **for** $n = 1, 2, \ldots$ **do**
>   Skeptic announces $M_n \geq 0$.
>   Reality announces $x_n \in [-1, \infty)$.
> $\mathcal{K}_n = \mathcal{K}_{n-1} + M_n x_n$
> **end for**

(Here *lower bounded* refers to the outcomes satisfying $x_n \geq -1$, while *upper forecasting* refers to $M_n \geq 0$, which is appropriate for testing whether the price of zero for x_n is too high.)

Adapt the proof of Lemma 3.3 to show that Skeptic can force

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^n x_i^2 < \infty \;\Rightarrow\; \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^n x_i \;\leq\; 0,$$

(where $E \Rightarrow F$ denotes the event $E^c \cup F$).

2 Chapter 4

The next two questions are for the Unbounded Forecasting Game (with means set to zero $m_n = 0$), which has protocol

> $\mathcal{K}_0 = 1.$
> **for** $n = 1, 2, \ldots$ **do**
>   Forecaster announces $v_n \geq 0$.
>   Skeptic announces $M_n \in \mathbb{R}$ and $V_n \geq 0$.
>   Reality announces $x_n \in \mathbb{R}$.
> $\mathcal{K}_n = \mathcal{K}_{n-1} + M_n x_n + V_n\big(x_n^2 - v_n\big)$
> **end for**

Exercise 4:

Let $\mathcal{T}$ be a nonnegative supermartingale, and let $\mathcal{P}$ be a bounding strategy for $\mathcal{T}$ (i.e. a strategy witnessing a bounding capital process). As in the proof of Lemma 4.5, fix $0 < a < b$, with the goal of getting rich if $\mathcal{T}$ keeps oscillating below a and above b forever. Set $\tau_0 = 0$ and for $k = 1, 2, \ldots$ set

$$\sigma_k \;=\; \min\{i > \tau_{k-1} \mid \mathcal{T}_i > b\} \qquad \text{and} \qquad \tau_k \;=\; \min\{i > \sigma_k \mid \mathcal{T}_i < a\},$$

so that $0 < \sigma_1 < \tau_1 < \sigma_2 \ldots$ are the times where $\mathcal{T}$ first passes above b and below a alternately. Let $\mathcal{P}^{a,b}$ be the strategy given by

$$\mathcal{P}_i^{a,b} \;=\; \begin{cases} \mathcal{P}_i & \text{if } \exists k : \tau_{k-1} < i \leq \sigma_k \\ (0,0) & \text{o.w.} \end{cases},$$

i.e. alternately gamble like $\mathcal{P}_i$ and do not gamble at all. Let $\mathcal{T}^{a,b} = \mathcal{T}(\square) + \mathcal{K}^{\mathcal{P}^{a,b}}$.

1. Show that $\mathcal{T}^{a,b}$ is a nonnegative supermartingale.
2. Show that $\lim_{n\to\infty} \mathcal{T}_n^{a,b} = \infty$ whenever

$$\liminf_{n\to\infty} \mathcal{T}_n < a \qquad \text{and} \qquad b < \limsup_{n\to\infty} \mathcal{T}_n.$$

Exercise 5:

Fix any $c \in \mathbb{R}$. Show that $\mathcal{U}_n = \left(c + \sum_{i=1}^n \frac{x_i}{i}\right)^2$ is a semimartingale by finding a supermartingale $\mathcal{T}$ and compensator $\mathcal{A}$ such that $\mathcal{U} = \mathcal{T} + \mathcal{A}$.