Game Theoretic Probability Homework 1

Monday 30^{th} May, 2016

This set is due on Wednesday 1^{st} June, 2016 before class.
All protocols are *coherent* unless specified.

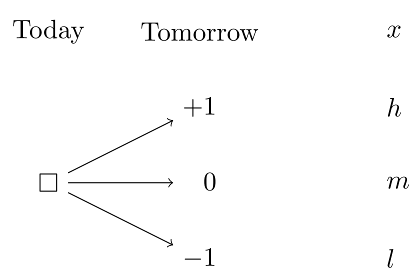
Exercise 1:
Fix variables $x, y : \Omega \to \mathbb{R}$.

1. Show the duality relationship $\overline{\mathbb{E}}[x] = -\underline{\mathbb{E}}[-x]$.
2. Show that $\overline{\mathbb{E}}[x + y] \leq \overline{\mathbb{E}}[x] + \overline{\mathbb{E}}[y]$.
3. Show that $\mathbb{E}[x + y] = \mathbb{E}[x] + \mathbb{E}[y]$ (assuming all these exist).

Exercise 2:
Fix any event $E \subseteq \Omega$ with complement $E^c = \Omega \setminus E$. Show

1. $0 \leq \underline{\mathbb{P}}(E) \leq \overline{\mathbb{P}}(E) \leq 1$.
2. $\underline{\mathbb{P}}(E) = 1 - \overline{\mathbb{P}}(E^c)$

Exercise 3:
Consider the following one-round protocol with sample space $\Omega = \{-1, 0, +1\}$.



Assume Skeptic can buy, at no cost, any multiple $M \in \mathbb{R}$ of the gamble that pays y when Reality moves to $y \in \Omega$, increasing his capital by My. Consider the variable x, defined in terms of arbitrary payoffs $h, m, l \in \mathbb{R}$.

1. Compute $\overline{\mathbb{E}}[x]$ and $\underline{\mathbb{E}}[x]$.
2. For h, m, l such that $\overline{\mathbb{E}}[x] = \underline{\mathbb{E}}[x]$, also compute $\overline{\mathbb{V}}[x]$ and $\underline{\mathbb{V}}[x]$.

Exercise 4:

Consider a one-round symmetric protocol with sample space $\Omega = [-1, 1]$. Reality chooses the outcome $y \in [-1, 1]$. Before seeing y, Skeptic may take any real number of tickets paying $y \in [-1, 1]$ at price 0. That is, if he buys $M \in \mathbb{R}$ tickets, his capital increases by My. Consider a variable $f : [-1, 1] \to \mathbb{R}$ (you may assume f is differentiable). Compute $\overline{\mathbb{E}}[f]$ and give the witnessing strategy for Skeptic in case

1. f is convex,
2. f is concave.

(Hint: check that your answers agree for linear f, which satisfy both cases.)

Exercise 5:

Prove the following claim: If Skeptic can force $E \in \Omega$ then $\overline{\mathbb{P}}(E^c) = 0$.